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LECTURES ON QUASI-INVARIANTS OF COXETER GROUPS AND THE CHEREDNIK ALGEBRA

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INTRODUCTION

This paper arose from a series of three lectures given by the first author at the Università di Roma "Tor Vergata" in January 2002, when the second author extended and improved her notes of these lectures. It contains an elementary introduction for non-specialists to the theory of quasi-invariants (but no original results).

Our main object of study is the variety X_m of quasi-invariants for a finite Coxeter group. This very interesting singular algebraic variety arose in work of O. Chalykh and A. Veselov about 10 years ago, as the spectral variety of the quantum Calogero-Moser system. We will see that despite being singular, this variety has very nice properties (Cohen-Macaulay, Gorenstein, simplicity of the ring of differential operators, explicitly given Hilbert series). One should remark that although the definition of X_m is completely elementary, it is helpful, in order to understand the geometry of X_m, to use representation theory of the rational degeneration of Cherednik's double affine Hecke algebra, and the theory of integrable systems. Thus, the study of X_m leads us to a junction of three subjects — integrable systems, representation theory, and algebraic geometry. The content of the paper is as follows. In Lecture 1 we define the ring of quasi-invariants for a Coxeter group, and discuss its elementary properties (with proofs), as well as deeper properties, such as Cohen-Macaulay, the Gorenstein property, and the Hilbert series (whose partial

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proofs are postponed until Lecture 3). In Lecture 2, we explain the origin of the ring of quasi-invariants in the theory of integrable systems, and introduce some tools from integrable systems, such as the Baker-Akhieser function. Finally, in Lecture 3, we develop the theory of the rational Cherednik algebra, the representation-theoretic techniques due to Opdam and Rouquier, and finish the proofs of the geometric statements from Chapter 1.

1. Lecture 1

1.1 Definition of quasi-invariants

In this lecture we will define the ring of quasi-invariants Q_m and discuss its main properties.

We will work over the field $\mathbf{C}$ of complex numbers. Let W be a finite Coxeter group, i.e. a finite group generated by reflections. Let us denote by $\mathfrak{h}$ its reflection representation. A typical example is the Weyl group of a semisimple Lie algebra acting on a Cartan subalgebra $\mathfrak{h}$. In the case the Lie algebra is $\mathfrak{sl}(n)$, we have that W is the symmetric group S_n on n letters and $\mathfrak{h}$ is the space of diagonal traceless $n \times n$ matrices.

Let $\Sigma \subset W$ denote the set of reflections. Clearly, W acts on Σ by conjugation. Let $m\colon \Sigma \to \mathbf{Z}_+$ be a function on Σ taking non negative integer values, which is W-invariant. The number of orbits of W on Σ is generally very small. For example, if W is the Weyl group of a simple Lie algebra of ADE type, then W acts transitively on Σ, so m is a constant function.

For each reflection $s \in \Sigma$, choose $\alpha_s \in \mathfrak{h}^* - \{0\}$ so that, for $x \in \mathfrak{h}$, $\alpha_s(sx) = -\alpha_s(x)$ (this means that the hyperplane given by the equation $\alpha_s = 0$ is the reflection hyperplane for s).

Definition 1.1 ([CV1, CV2]). A polynomial $q \in \mathbf{C}[\mathfrak{h}]$ is said to be *m-quasi-invariant* with respect to W if, for any $s \in \Sigma$, the polynomial $q(x) - q(sx)$ is divisible by $\alpha_s(x)^{2m_s+1}$.

We will denote by Q_m the space of m-quasi-invariant polynomials with respect to W.

Notice that every element of $\mathbf{C}[\mathfrak{h}]$ is a 0-quasi-invariant, and that every W-invariant is an m-quasi-invariant for any m. Indeed if $q \in \mathbf{C}[\mathfrak{h}]^W$, then we have $q(x) - q(sx) = 0$ for all $s \in \Sigma$, and 0 is divisible by all powers of $\alpha_s(x)$. Thus in a way, $\mathbf{C}[\mathfrak{h}]^W$ can be viewed as the set of ∞-quasi-invariants.

EXAMPLE 1.2. The group $W = \mathbf{Z}/2$ acts on $\mathfrak{h} = \mathbf{C}$ by $s(v) = -v$. In this case m is a non negative integer and $\Sigma = \{s\}$. So this definition says that q is in Q_m iff $q(x) - q(-x)$ is divisible by x^{2m+1}. It is very easy to write a basis of Q_m. It is given by the polynomials $\{x^{2i} \mid i \geq 0\} \cup \{x^{2i+1} \mid i \geq m\}$.

1.2 ELEMENTARY PROPERTIES OF Q_m

Some elementary properties of Q_m are collected in the following proposition.

PROPOSITION 1.3 (see [FV] and references therein).

1) $\mathbf{C}[\mathfrak{h}]^W \subset Q_m \subseteq \mathbf{C}[\mathfrak{h}]$, $Q_0 = \mathbf{C}[\mathfrak{h}]$, $Q_m \subset Q_{m'}$ *if* $m \geq m'$, $\bigcap_m Q_m = \mathbf{C}[\mathfrak{h}]^W$.
2) Q_m *is a graded subalgebra of* $\mathbf{C}[\mathfrak{h}]$.
3) *The fraction field of* Q_m *is equal to* $\mathbf{C}(\mathfrak{h})$.
4) Q_m *is a finite* $\mathbf{C}[\mathfrak{h}]^W$*-module and a finitely generated algebra.* $\mathbf{C}[\mathfrak{h}]$ *is a finite* Q_m*-module.*

Proof. 1) is immediate and has already been mentioned in 1.1.

2) Clearly Q_m is closed under addition. Let $p, q \in Q_m$. Let $s \in \Sigma$. Then

$$p(x)q(x) - p(sx)q(sx) = (p(x) - p(sx))q(x) + p(sx)(q(x) - q(sx)).$$

Since both $p(x) - p(sx)$ and $q(x) - q(sx)$ are divisible by $\alpha_s^{2m_s+1}$, we deduce that $p(x)q(x) - p(sx)q(sx)$ is also divisible by $\alpha_s^{2m_s+1}$, proving the claim.

3) Consider the polynomial

$$\delta_{2m+1}(x) = \prod_{s \in \Sigma} \alpha_s(x)^{2m_s+1}.$$

This polynomial is uniquely defined up to scaling. One has $\delta_{2m+1}(sx) = -\delta_{2m+1}(x)$ for each $s \in \Sigma$, hence $\delta_{2m+1} \in Q_m$. Take $f(x) \in \mathbf{C}[\mathfrak{h}]$. We claim that $f(x)\delta_{2m+1}(x) \in Q_m$. As a matter of fact,

$$f(x)\delta_{2m+1}(x) - f(sx)\delta_{2m+1}(sx) = (f(x) + f(sx))\delta_{2m+1}(x),$$

and by its definition $\delta_{2m+1}(x)$ is divisible by $\alpha_s(x)^{2m_s+1}$ for all $s \in \Sigma$. This implies 3).

4) By Hilbert's theorem on the finiteness of invariants, we get that $\mathbf{C}[\mathfrak{h}]^W$ is a finitely generated algebra over $\mathbf{C}$ and $\mathbf{C}[\mathfrak{h}]$ is a finite $\mathbf{C}[\mathfrak{h}]^W$-module and hence a finite Q_m-module, proving the second part of 4).

Now $Q_m \subset \mathbf{C}[\mathfrak{h}]$ is a submodule of the finite module $\mathbf{C}[\mathfrak{h}]$ over the Noetherian ring $\mathbf{C}[\mathfrak{h}]^W$. Hence it is finite. This immediately implies that Q_m is a finitely generated algebra over $\mathbf{C}$. □

REMARK. In fact, since W is a finite Coxeter group, a celebrated result of Chevalley says that the algebra $\mathbf{C}[\mathfrak{h}]^W$ is not only a finitely generated $\mathbf{C}$-algebra but actually a free (=polynomial) algebra. Namely, it is of the form $\mathbf{C}[q_1, \dots, q_n]$, where the q_i are homogeneous polynomials of some degrees d_i. Furthermore, if we denote by H the subspace of $\mathbf{C}[\mathfrak{h}]$ of harmonic polynomials, i.e. of polynomials killed by W-invariant differential operators with constant coefficients without constant term, then the multiplication map

$$\mathbf{C}[\mathfrak{h}]^W \otimes H \to \mathbf{C}[\mathfrak{h}]$$

is an isomorphism of $\mathbf{C}[\mathfrak{h}]^W$- and of W-modules. In particular, $\mathbf{C}[\mathfrak{h}]$ is a free $\mathbf{C}[\mathfrak{h}]^W$-module of rank $|W|$.

1.3 THE VARIETY X_m AND ITS BIJECTIVE NORMALIZATION

Using Proposition 1.3, we can define the irreducible affine variety $X_m = \mathrm{Spec}(Q_m)$. The inclusion $Q_m \subset \mathbf{C}[\mathfrak{h}]$ induces a morphism

$$\pi : \mathfrak{h} \to X_m ,$$

which again by Proposition 1.3 is birational and surjective. (Notice that in particular this implies that X_m is singular for all $m \neq 0$.)

In fact, not only is π birational, but a stronger result is true.

PROPOSITION 1.4 (Berest, see [BEG]). *π is a bijection.*

Proof. By the above remarks, we only have to show that π is injective. In order to achieve this, we need to prove that quasi-invariants separate points of $\mathfrak{h}$, i.e. that if $z, y \in \mathfrak{h}$ and $z \neq y$, then there exists $p \in Q_m$ such that $p(z) \neq p(y)$. This is obtained in the following way. Let $W_z \subset W$ be the stabilizer of z and choose $f \in \mathbf{C}[\mathfrak{h}]$ such that $f(z) \neq 0$, $f(y) = 0$. Set

$$p(x) = \prod_{s \in \Sigma, sz \neq z} \alpha_s(x)^{2m_s+1} \prod_{w \in W_z} f(wx) .$$

We claim that $p(x) \in Q_m$. Indeed, let $s \in \Sigma$ and assume that $s(z) \neq z$.

We have by definition $p(x) = \alpha_s(x)^{2m_s+1}\tilde{p}(x)$, with $\tilde{p}(x)$ a polynomial. So

$$p(x) - p(sx) = \alpha_s(x)^{2m_s+1}\tilde{p}(x) - \alpha_s(sx)^{2m_s+1}\tilde{p}(sx) = \alpha_s(x)^{2m_s+1}(\tilde{p}(x) + \tilde{p}(sx)) .$$

If on the other hand, $sz = z$, i.e. $s \in W_z$, then s preserves the set $W \setminus W_z$, and hence preserves $\prod_{s \in \Sigma \cap (W \setminus W_z)} \alpha_s(x)^{2m_s+1}$ (as it acts by -1 on the products $\prod_{s \in \Sigma} \alpha_s(x)^{2m_s+1}$ and $\prod_{s \in \Sigma \cap W_z} \alpha_s(x)^{2m_s+1}$). Since $\prod_{w \in W_z} f(wx)$ is

W_z-invariant, we deduce that $p(x)-p(sx)=0$, so that in this case $p(x)-p(sx)$ also is divisible by $\alpha_s(x)^{2m_s+1}$.

To conclude, notice that $p(z)\neq 0$. Indeed, for a reflection s, α_s vanishes exactly on the fixed points of s, so that $\prod_{s\in\Sigma, sz\neq z}\alpha_s(z)^{2m_s+1}\neq 0$. Also for all $w\in W_z$ $f(wz)=f(z)\neq 0$. On the other hand, it is clear that $p(y)=0$. □

EXAMPLE 1.5. Take $W=\mathbf{Z}/2$. As we have already seen, Q_m has a basis given by the monomials $\{x^{2i}\mid i\geq 0\}\cup\{x^{2i+1}\mid i\geq m\}$. From this we deduce that setting $z=x^2$ and $y=x^{2m+1}$, $Q_m=\mathbf{C}[y,z]/(y^2-z^{2m+1})=\mathbf{C}[K]$, where K is the plane curve with a cusp at the origin, given by the equation $y^2=z^{2m+1}$. The map $\pi\colon \mathbf{C}\to K$ is given by $\pi(t)=(t^{2m+1},t^2)$, which is clearly bijective.

1.4 FURTHER PROPERTIES OF X_m

Let us get to some deeper properties of quasi-invariants. Let X be an irreducible affine variety over $\mathbf{C}$ and $A=\mathbf{C}[X]$. Recall that, by the Noether Normalization Lemma, there exist $f_1,\dots,f_n\in\mathbf{C}[X]$ which are algebraically independent over $\mathbf{C}$ and such that $\mathbf{C}[X]$ is a finite module over the polynomial ring $\mathbf{C}[f_1,\dots,f_n]$. This means that we have a finite morphism of X onto an affine space.

DEFINITION 1.6. A (and X) is said to be *Cohen-Macaulay* if there exist $f_1,\dots,f_n$ as above, with the property that $\mathbf{C}[X]$ is a locally free module over $\mathbf{C}[f_1,\dots,f_n]$. (Notice that by the Quillen-Suslin theorem, this is equivalent to saying that A is a free module.)

REMARK. If A is Cohen-Macaulay, then for any $f_1,\dots,f_n$ which are algebraically independent over $\mathbf{C}$ and such that A is a finite module over the polynomial ring $\mathbf{C}[f_1,\dots,f_n]$, we have that A is a locally free $\mathbf{C}[f_1,\dots,f_n]$-module, see [Eis], Corollary 18.17.

THEOREM 1.7 ([EG2], [BEG], conjectured in [FV]). *Q_m is Cohen-Macaulay.*

Notice that, using Chevalley's result that $\mathbf{C}[\mathfrak{h}]^W$ is a polynomial ring, it will suffice, in order to prove Theorem 1.7, to prove:

THEOREM 1.8 ([EG2, BEG], conjectured in [FV]). *Q_m is a free $\mathbf{C}[\mathfrak{h}]^W$-module.*

We show how one can prove this Theorem in 3.10. This proof follows [BEG] (the original proof of [EG2] is shorter but somewhat less conceptual). The main idea of the proof is to show that the $\mathbf{C}[\mathfrak{h}]^W$-module Q_m can be extended to a module over a bigger (noncommutative) algebra, namely the spherical subalgebra of the rational Cherednik algebra. Furthermore, this module belongs to an appropriate category of representations of this algebra, called category $\mathcal{O}$. On the other hand, it can be shown that any module over the spherical subalgebra that belongs to this category is free when restricted to the commutative algebra $\mathbf{C}[\mathfrak{h}]^W$.

1.5 The Poincaré series of Q_m

Consider now the Poincaré series

$$h_{Q_m}(t) = \sum_{r \geq 0} \dim Q_m[r] t^r ,$$

where $Q_m[r]$ denotes the graded component of Q_m of degree r. For every irreducible representation $\tau \in \widehat{W}$, define

$$\chi_\tau(t) = \sum_{r \geq 0} \dim \mathrm{Hom}_W(\tau, \mathbf{C}[\mathfrak{h}][r]) t^r .$$

Consider the element in the group ring $\mathbf{Z}[W]$

$$\mu_m = \sum_{s \in \Sigma} m_s (1 - s) .$$

The W-invariance of m implies that μ_m lies in the center of $\mathbf{Z}[W]$. Hence it is clear that μ_m acts as a scalar, $\xi_m(\tau)$, on τ. Let d_τ be the degree of τ.

LEMMA 1.9. *The scalar $\xi_m(\tau)$ is an integer.*

Proof. $\mathbf{Z}[W]$ and hence also its center, is a finite $\mathbf{Z}$-module. This clearly implies that $\xi_m(\tau)$ is an algebraic integer. Thus to prove that $\xi_m(\tau)$ is an integer, it suffices to see that $\xi_m(\tau)$ is a rational number. Let $d_{\tau,s}$ be the dimension of the space of s-invariants in τ. Taking traces we get

$$d_\tau \xi_m(\tau) = \sum_{s \in \Sigma} 2 m_s (d_\tau - d_{\tau,s}) ,$$

which gives the rationality of $\xi_m(\tau)$. □

THEOREM 1.10. *One has*

$$h_{Q_m}(t) = \sum_{\tau \in \widehat{W}} d_\tau t^{\xi_m(\tau)} \chi_\tau(t) . \tag{1}$$

REMARK. This theorem was proved in [FeV] modulo Theorem 1.7 (conjectured in [FV]) using the so-called Matsuo-Cherednik correspondence (see [FeV] for details). Thus, Theorem 1.10 follows from [FeV] and [EG2]. Another proof of this theorem is given in [BEG]; this is the proof we will discuss below (in Lecture 3).

EXAMPLE 1.11. If $m = 0$, since $Q_0 = \mathbf{C}[\mathfrak{h}]$, the theorem says that

$$h_{Q_0}(t) = \frac{1}{(1-t)^n} = \sum_{\tau \in \widehat{W}} d_\tau \chi_\tau(t) .$$

Indeed, as a W-module one has

$$\mathbf{C}[\mathfrak{h}] = \oplus_\tau \tau \otimes \mathrm{Hom}_W(\tau, \mathbf{C}[\mathfrak{h}]) .$$

EXAMPLE 1.12. If $W = \mathbf{Z}/2$, then $\widehat{W} = \{+, -\}$, where $+$ (respectively $-$) denotes the trivial (respectively the sign) representation. One has

$$\mathbf{C}[x] = \mathbf{C}[x^2] \oplus \mathbf{C}[x^2]x ,$$

where $\mathbf{C}[x^2] = \mathbf{C}[x]^W$ and $\mathbf{C}[x^2]x$ is the isotypic component of the sign representation. Thus

$$\chi_+(t) = \frac{1}{1-t^2} , \quad \chi_-(t) = \frac{t}{1-t^2} ,$$

$\mu_m = m(1-s)$. Thus $\xi_m(+) = 0$, $\xi_m(-) = 2m$. We deduce that

$$h_{Q_m}(t) = \frac{1}{1-t^2} + \frac{t^{2m+1}}{1-t^2} ,$$

as we already know.

Recall now that as a graded W-module $\mathbf{C}[\mathfrak{h}]$ is isomorphic to $\mathbf{C}[\mathfrak{h}]^W \otimes H$, H being the space of harmonic polynomials. We deduce that the τ-isotypic component in $\mathbf{C}[\mathfrak{h}]$ is isomorphic to $\mathbf{C}[\mathfrak{h}]^W \otimes H_\tau$.

Set $K_\tau(t) = \sum_{r\geq 0} \dim \mathrm{Hom}_W(\tau, H[r])t^r$. This is a polynomial, called the Kostka polynomial relative to τ. We deduce that

$$\chi_\tau(t) = \frac{K_\tau(t)}{\prod_{i=1}^n (1-t^{d_i})}. \tag{2}$$

Also, if $\tau' = \tau \otimes \varepsilon$, ε being the sign representation, one has

$$K_{\tau'}(t) = K_\tau(t^{-1})t^{|\Sigma|}.$$

Set now

$$P_m(t) = \sum_{\tau\in\widehat{W}} d_\tau t^{\xi_m(\tau)} K_\tau(t).$$

We have

PROPOSITION 1.13 ([FeV]).

$$h_{Q_m}(t) = \frac{P_m(t)}{\prod_{i=1}^n (1-t^{d_i})}.$$

Furthermore $P_m(t) = t^{\xi_m(\varepsilon)+|\Sigma|}P_m(t^{-1})$.

Proof. Substituting the expression (2) for $\chi_\tau(t)$ in (1.10) and using the definition of $P_m(t)$, we get

$$h_{Q_m}(t) = \sum_{\tau\in\widehat{W}} d_\tau t^{\xi_m(\tau)} \frac{K_\tau(t)}{\prod_{i=1}^n (1-t^{d_i})} = \frac{P_m(t)}{\prod_{i=1}^n (1-t^{d_i})},$$

as desired.

Now notice that

$$\xi_m(\tau) + \xi_m(\tau') = \sum_{s\in\Sigma} 2m_s = \xi_m(\varepsilon).$$

Using this we get

$$\begin{aligned} t^{\xi_m(\varepsilon)+|\Sigma|}P_m(t^{-1}) &= \sum_{\tau\in\widehat{W}} d_\tau t^{\xi_m(\varepsilon)-\xi_m(\tau)}t^{|\Sigma|}K_\tau(t^{-1}) \\ &= \sum_{\tau'\in\widehat{W}} d_{\tau'} t^{\xi_m(\tau')}K_{\tau'}(t) = P_m(t), \end{aligned}$$

as desired. □

From this we deduce

THEOREM 1.14 ([EG2, BEG, FeV], conjectured in [FV]). *The ring Q_m of m-quasi-invariants is Gorenstein.*

Proof. By Stanley's theorem (see [Eis]), a positively graded Cohen-Macaulay domain A is Gorenstein iff its Poincaré series is a rational function $h(t)$ satisfying the equation $h(t^{-1}) = (-1)^n t^l h(t)$, where l is an integer and n is the dimension of the spectrum of A. Thus the result follows immediately from Proposition 1.13. □

1.6 THE RING OF DIFFERENTIAL OPERATORS ON X_m

Finally, let us introduce the ring $\mathcal{D}(X_m)$ of differential operators on X_m, that is the ring of differential operators with coefficients in $\mathbf{C}(\hbar)$ mapping Q_m to Q_m. It is clear that this definition coincides with Grothendieck's well-known definition ([Bj]).

THEOREM 1.15 ([BEG]). *$\mathcal{D}(X_m)$ is a simple algebra.*

REMARK 1.16. a) The ring of differential operators on a smooth affine algebraic variety is always simple (see [Bj], Chapter 3).

b) By a result of M. van den Bergh [VdB], for a non-smooth variety, the simplicity of the ring of differential operators implies the Cohen-Macaulay property of this variety.

2. LECTURE 2

We will now see how the ring Q_m appears in the theory of completely integrable systems.

2.1 HAMILTONIAN MECHANICS AND INTEGRABLE SYSTEMS

Recall the basic setup of Hamiltonian mechanics [Ar]. Consider a mechanical system with configuration space X (a smooth manifold). Then the phase space of this system is T^*X, the cotangent bundle on X. The space T^*X is naturally a symplectic manifold, and in particular we have an operation of Poisson bracket on functions on T^*X. A point of T^*X is a pair (x,p), where $x \in X$ is the position and $p \in T_x^*X$ is the momentum. Such pairs are

called states of the system. The dynamics of the system $x = x(t)$, $p = p(t)$ depends on the Hamiltonian, or energy function, $E(x,p)$ on T^*X. Given E and the initial state $x(0)$, $p(0)$, one can recover the dynamics $x = x(t)$, $p = p(t)$ from Hamilton's differential equations $\frac{df(x,p)}{dt} = \{f,E\}$. If X is locally identified with $\mathbf{R}^n$ by choosing coordinates $x_1,\dots,x_n$, then T^*X is locally identified with $\mathbf{R}^{2n}$ with coordinates $x_1,\dots,x_n,p_1,\dots,p_n$. In these coordinates, Hamilton's equations may be written in their standard form

$$\dot{x}_i = \frac{\partial E}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial E}{\partial x_i}.$$

A function $I(x,p)$ is called an integral of motion for our system if $\{I,E\} = 0$. Integrals of motion are useful, since for any such integral I the function $I(x(t),p(t))$ is constant, which allows one to reduce the number of variables by 2. Thus, if we are given n functionally independent integrals of motion $I_1,\dots,I_n$ with $\{I_l,I_k\} = 0$ for all $1 \le l,k \le n$, then all $2n$ variables x_i,p_i can be excluded, and the system can be completely solved by quadratures. Such a situation is called complete (or Liouville) integrability.

2.2 The classical Calogero-Moser system

Quasi-invariants are related to many-particle systems. Consider a system of n particles on the real line $\mathbf{R}$. A potential is an even function

$$U(x) = U(-x), \quad x \in \mathbf{R}.$$

Two particles at points a, b have energy of interaction $U(a-b)$. The total energy of our system of particles is

$$E = \sum_{i=1}^{n} \frac{p_i^2}{2} + \sum_{i<j} U(x_i - x_j).$$

Here, x_i are the coordinates of the particles, p_i their momenta. The dynamics of the particles $x_i = x_i(t)$, $p_i = p_i(t)$ is governed by the Hamilton equations with energy function E.

This is a system of nonlinear differential equations, which in general can be difficult to solve explicitly. However, for special potentials this system might be completely integrable. For instance, we will see that this is the case for the Calogero-Moser potential,

$$U(x) = \frac{\gamma}{x^2},$$

γ being a constant.

The Calogero-Moser system has a generalization to arbitrary Coxeter groups. Namely, consider a finite group W generated by reflections acting on the space $\mathfrak{h}$, and keep the notation of the previous section. Fix a W-invariant nondegenerate scalar product $(-,-)$ on $\mathfrak{h}$. It determines a scalar product on $\mathfrak{h}^*$. Define the "energy function"

$$E(x,p) = \frac{(p,p)}{2} + \frac{1}{2}\sum_{s\in\Sigma} \frac{\gamma_s(\alpha_s,\alpha_s)}{\alpha_s(x)^2}, \quad x \in \mathfrak{h}, \quad p \in \mathfrak{h}^*$$

on $T^*\mathfrak{h} = \mathfrak{h} \times \mathfrak{h}^*$, where $\gamma\colon \Sigma \to \mathbf{C}$ is a W-invariant function. Notice that although α_s is defined up to a non zero constant, by homogeneity, E is independent of the choice of α_s. We will call the system defined by E the Calogero-Moser system for W.

If W is the symmetric group S_n, $\mathfrak{h} = \mathbf{C}^n$, then Σ is the set of transpositions $s_{i,j}$, $i < j$, and we can take $\alpha_s = e_i - e_j$, Then we clearly obtain the usual Calogero-Moser system.

Below we will see that the Calogero-Moser system for W is completely integrable.

2.3 The quantum Calogero-Moser system

Let us now discuss quantization of the Calogero-Moser system. We start by quantizing the energy E by formally making the substitution

$$p_j \Rightarrow -i\hbar\frac{\partial}{\partial x_j},$$

where $\hbar$ is a parameter (Planck's constant). This yields the Schrödinger operator

$$\widehat{E} := -\frac{\hbar^2}{2}\Delta + \frac{1}{2}\sum_{s\in\Sigma} \frac{\gamma_s(\alpha_s,\alpha_s)}{\alpha_s^2},$$

where Δ denotes the Laplacian.

In particular, in the case of $W = S_n$ we have

$$\widehat{E} = -\frac{\hbar^2}{2}\Delta + \sum_{i<j} \frac{c}{(x_i - x_j)^2},$$

where $\Delta = \sum_i \frac{\partial^2}{\partial x_i^2}$. Setting $\beta_s = \frac{\gamma_s}{2\hbar^2}$, we will from now on consider the operator

$$H := -\frac{2}{\hbar^2}\widehat{E} = \Delta - \sum_{s\in\Sigma} \frac{\beta_s(\alpha_s,\alpha_s)}{\alpha_s^2(x)},$$

called the Calogero-Moser operator.

We want to study the stationary Schrödinger equation:

$$H\psi = \lambda\psi, \quad \lambda \in \mathbf{C}. \tag{3}$$

As in the classical case, it is difficult to say anything explicit about solutions of this equation for a general Schrödinger operator H, but for the Calogero-Moser operator the situation is much better.

DEFINITION 2.1. A *quantum integral* of H is a differential operator M such that

$$[M, H] = 0.$$

We are going to show that there are many quantum integrals of H, namely that there are n commuting algebraically independent quantum integrals $M_1, \ldots, M_n$ of H. By definition, this means that the quantum Calogero-Moser system is completely integrable.

Once we have found $M_1, \ldots, M_n$, observe that for fixed constants $\mu_1, \ldots, \mu_n$, the space of solutions of the system

$$\begin{cases} M_1\psi = \mu_1\psi \\ \quad \ldots\ldots \\ M_n\psi = \mu_n\psi \end{cases}$$

is clearly stable under H. We will see that this space is in fact finite dimensional. Therefore, the operators M_i allow one to reduce the problem of solving the partial differential equation $H\psi = \lambda\psi$ to that of solving a system of ordinary linear differential equations. This phenomenon is called quantum complete integrability.

2.4 THE ALGEBRA OF DIFFERENTIAL-REFLECTION OPERATORS .

We are now going to explain how to find quantum integrals for H, using the Dunkl-Cherednik method.

First let us fix some notation. Given a smooth affine variety X, we will denote by $\mathcal{D}(X)$ the ring of differential operators on X. We are going to consider the case in which X is the open set U in $\mathfrak{h}$ which is the complement of the divisor of the equation $\delta(x) := \prod_{s\in\Sigma} \alpha_s(x)$. Clearly $\mathcal{D}(U) = \mathcal{D}(\mathfrak{h})[1/\delta(x)]$.

LEMMA 2.2. *An element of $\mathcal{D}(U)$ is completely determined by its action on $\mathbf{C}[U]^W = \mathbf{C}[U/W]$.*

Proof. Recall that the quotient map $\pi\colon U \to U/W$ is finite and unramified. This implies that

$$\mathcal{D}(U) = \mathbf{C}[U] \otimes_{\mathbf{C}[U/W]} \mathcal{D}(U/W)\,.$$

From this we obtain that if $P \in \mathcal{D}(U)$ is such that $Pf = 0$ for all $f \in \mathbf{C}[U/W]$, then $P = 0$. □

We also have the operators on $\mathbf{C}[U]$ given by the action of W. We will denote by $\mathcal{A}$ the algebra of operators on U generated by $\mathcal{D}(U)$ and W, and call it the algebra of differential-reflection operators. The action of W on U induces an action on $\mathcal{D}(U)$, so that the subalgebra $\mathcal{D}(U) \subset \mathcal{A}$ is preserved by conjugation by elements of W. We have:

PROPOSITION 2.3. *$\mathcal{A} = \mathcal{D}(U) \rtimes W$, i.e. every element in $A \in \mathcal{A}$ can be uniquely written as a linear combination*

$$A = \sum_{w \in W} P_w w$$

with $P_w \in \mathcal{D}(U)$.

Proof. The fact that every element in $\mathcal{A}$ can be expressed as a linear combination $\sum_{w\in W} P_w w$ is clear. To show that such an expression is unique, assume $\sum_{w\in W} P_w w = 0$. Take $f \in \mathbf{C}[U]$ such that ${}^w f \neq {}^u f$ for all $w \neq u$ in W, and multiply the operator $\sum P_w w$ on the right by the operator of multiplication by the function f^i, $i \geq 0$. Then we get

$$\sum_{w\in W} P_w \circ ({}^w f)^i w = \sum_{w\in W} P_w w \circ f^i = 0\,.$$

Applying both sides of this equation to a function $g \in \mathbf{C}[U/W]$ we have

$$\sum_{w\in W} (P_w \circ {}^w f^i)\, g = 0\,.$$

Thus by Lemma 2.2, $\sum_{w\in W} P_w \circ {}^w f^i = 0$ for all i. Therefore, by Vandermonde's determinant formula, $P_w \circ \prod_{w\neq u}({}^w f - {}^u f) = 0$ and hence $P_w = 0$, for all $w \in W$, as desired. □

Take $A \in \mathcal{A}$ and write

$$A = \sum_{w \in W} P_w w\,.$$

We set $m(A) = \sum_{w\in W} P_w \in \mathcal{D}(U)$. Notice that if f is a W-invariant function, then clearly $A(f) = m(A)(f)$ and that, by what we have seen in Lemma 2.2, $m(A)$ is completely determined by its action on invariant functions.

In general, m is not a homomorphism. However:

PROPOSITION 2.4. *Let $\mathcal{A}^W \subset \mathcal{A}$ denote the subalgebra of elements invariant under conjugation by W. Then the restriction of m to $\mathcal{A}^W$ is an algebra homomorphism.*

Proof. If $A \in \mathcal{A}^W$, then clearly $m(A)$ is W-invariant. Now if we take $A, B \in \mathcal{A}^W$ and f a W-invariant function we have that $B(f)$ is also W-invariant. So

$$m(AB)(f) = (AB)(f) = A(B(f)) = A(m(B)(f)) = m(A)(m(B)(f)).$$

Thus $m(AB)$ and $m(A)m(B)$ coincide on W-invariant functions and hence coincide. □

2.5 DUNKL OPERATORS AND SYMMETRIC QUANTUM INTEGRALS

In this subsection we will construct quantum integrals of the Calogero-Moser operator. This construction is due to Heckman [He] and is based on the Dunkl operators, introduced in [Du].

Fix a W-invariant function $c: \Sigma \to \mathbf{C}$ such that $\beta_s = c_s(c_s + 1)$ for each $s \in \Sigma$. Set $\delta_c := \prod_{s\in\Sigma} \alpha_s(x)^{c_s}$ and define

$$L = \delta_c(x) H \delta_c(x)^{-1}.$$

Then an easy computation shows that

$$L = \Delta - \sum_{s\in\Sigma} \frac{2c_s}{\alpha_s(x)} \partial_{\alpha_s},$$

where, for a vector $y \in \mathfrak{h}$, the symbol ∂_y denotes, as usual, the partial derivative in the y direction (notice that using the scalar product we are viewing α_s as a vector in $\mathfrak{h}$ orthogonal to the hyperplane fixed by s).

From now on we will work with L instead of H and study the eigenvalue problem

$$L\psi = \lambda\psi. \tag{4}$$

It is clear that ψ is a solution of this equation if and only if $\delta_c(x)^{-1}\psi$ is a solution of (3).

Since for any $s \in \Sigma$ and $f \in \mathbf{C}[\mathfrak{h}]$ we have that $f(sx) - f(x)$ is divisible by $\alpha_s(x)$, the operator

$$\frac{1}{\alpha_s(x)}(s-1) \in \mathcal{A}$$

maps $\mathbf{C}[\mathfrak{h}]$ to itself.

DEFINITION 2.5. Given $y \in \mathfrak{h}$, we define the *Dunkl operator* D_y on $\mathbf{C}[\mathfrak{h}]$ by

$$D_y := \partial_y + \sum_{s \in \Sigma} c_s \frac{(\alpha_s, y)}{\alpha_s(x)}(s-1).$$

We have the following very important theorem.

THEOREM 2.6 ([Du]). *Let* $y, z \in \mathfrak{h}$. *Then*

$$[D_y, D_z] = 0.$$

Proof. See [Du], [Op]. □

PROPOSITION 2.7 (Heckman [He]). *Let* $\{y_1, \ldots, y_n\}$ *be an orthonormal basis of* $\mathfrak{h}$. *Then we have*

$$m(\sum_{i=1}^{n} D_{y_i}^2) = L.$$

Proof. Observe that $m(\sum_{i=1}^n D_{y_i}^2) = \sum_{i=1}^n m(D_{y_i}^2)$, so we need to compute $m(D_y^2)$ for $y \in \mathfrak{h}$. We have $m(D_y^2) = m(D_y m(D_y)) = m(D_y \partial_y)$. A simple computation shows that

$$D_y \partial_y = \partial_y^2 + \sum_{s \in \Sigma} c_s \frac{(\alpha_s, y)}{\alpha_s(x)}(\partial_y(s-1) - \frac{2(\alpha_s, y)}{(\alpha_s, \alpha_s)}\partial_{\alpha_s} s).$$

Thus

$$m(D_y^2) = \partial_y^2 - 2\sum_{s \in \Sigma} c_s \frac{(\alpha_s, y)^2}{(\alpha_s, \alpha_s)\alpha_s(x)}\partial_{\alpha_s}.$$

We get

$$m(\sum_{i=1}^{n} D_{y_i}^2) = \sum_i \partial_{y_i}^2 - 2\sum_{s \in \Sigma} c_s \frac{\sum_{i=1}^n (\alpha_s, y_i)^2}{(\alpha_s, \alpha_s)\alpha_s(x)}\partial_{\alpha_s} = L,$$

since $\sum_{i=1}^n (\alpha_s, y_i)^2 = (\alpha_s, \alpha_s)$. □

We are now ready to give the construction of quantum integrals of L. Consider the symmetric algebra $S\mathfrak{h} = \mathbf{C}[y_1, \ldots, y_n]$ which we can identify, using the fact that the Dunkl operators commute, with the polynomial ring $\mathbf{C}[D_{y_1}, \ldots, D_{y_n}] \subset \mathcal{A}$. The restriction of m to $S\mathfrak{h}^W$ is an algebra homomorphism into the ring $\mathcal{D}(U)$ (and in fact into $\mathcal{D}(U/W)$). Since $S\mathfrak{h}^W$ is itself a polynomial ring $\mathbf{C}[q_1, \ldots, q_n]$, with $q_1, \ldots, q_n$ of degree $d_1, \ldots, d_n$,

d_i being the degrees of basic W-invariants, we obtain a polynomial ring of commuting differential operators in $\mathcal{D}(U)$. Given $q \in \mathbf{C}[q_1, \dots, q_n]$ we will denote by L_q the corresponding differential operator. We may assume that $q_1 = \sum_{i=1}^n y_1^2$ so that $L = L_{q_1}$. Thus for every $q \in \mathbf{C}[q_1, \dots, q_n]$, L_q is a quantum integral of the quantum Calogero-Moser system. In particular, the operators $L_{q_1}, \dots, L_{q_n}$ are n algebraically independent pairwise commuting quantum integrals.

Now the eigenvalue problem (4) may be replaced by

$$L_p\psi = \lambda_p\psi$$

for $p \in \mathbf{C}[q_1, \dots, q_n]$, where the assignment $p \to \lambda_p$ is an algebra homomorphism $\mathbf{C}[q_1, \dots, q_n] \to \mathbf{C}$.

In other words, we may say that since $\mathbf{C}[q_1, \dots, q_n] = \mathbf{C}[\mathfrak{h}^*/W] = \mathbf{C}[\mathfrak{h}/W]$, for every point $k \in \mathfrak{h}/W$, we have the eigenvalue problem

$$L_p\psi = p(k)\psi\,. \tag{5}$$

PROPOSITION 2.8. *Near a generic point $x_0 \in \mathfrak{h}$, the system $L_p\psi = p(k)\psi$ has a space of solutions of dimension $|W|$.*

Proof. The proposition follows easily from the fact that the symbols of L_{q_i} are $q_i(\partial)$, and that $\mathbf{C}[y_1, \dots, y_n]$ is a free module over $\mathbf{C}[q_1, \dots, q_n]$ of rank $|W|$. □

2.6 ADDITIONAL INTEGRALS FOR INTEGER VALUED c

If $c_s \notin \mathbf{Z}$, the analysis of the solutions of the equations $L_p\psi = p(k)\psi$ is rather difficult (see [HO]). However, in the case $c\colon \Sigma \to \mathbf{Z}$, the system can be simplified. Let us consider this case. First remark that, since $\beta_s = c_s(c_s+1)$, by changing c_s to $-1-c_s$ if necessary, we may assume that c is non-negative. So we will assume that c takes non-negative integral values and we will denote it by m.

System (5) can be further simplified, if we can find a differential operator M (not a polynomial of $L_{q_1}, \dots, L_{q_n}$) such that $[M, L_p] = 0$ for all $p \in \mathbf{C}[q_1, \dots, q_n]$. Then the operator M will act on the space of solutions of (5), hopefully with distinct eigenvalues. So if μ is such an eigenvalue, the system

$$\begin{cases} L_p\psi = p(k)\,\psi \\ M\psi = \mu\,\psi \end{cases}$$

will have a one dimensional space of solutions and we can find the unique up to scaling solution ψ using Euler's formula.

Such an M exists if and only if $c = m$ has integer values. Namely, we will see that one can extend the homomorphism $\mathbf{C}[q_1, \dots, q_n] \to \mathcal{D}(U)$ mapping $q \to L_q$ to the ring of m-quasi-invariants Q_m.

We start by remarking that under some natural homogeneity assumptions, if such an extension exists, it is unique.

PROPOSITION 2.9. 1) *Assume that* $q \in \mathbf{C}[y_1, \dots, y_n]$ *is a homogeneous polynomial of degree* d. *If there exists a differential operator* M_q *with coefficients in* $\mathbf{C}(\mathfrak{h})$, *of the form*

$$M_q = q(\partial_{y_1}, \dots, \partial_{y_n}) + l.o.t.$$

such that $[M_q, L] = 0$, *whose homogeneity degree is* $-d$, *then* M_q *is unique.*

2) *Let* $\mathbf{C}[q_1, \dots, q_n] \subseteq B \subseteq \mathbf{C}[y_1, \dots, y_n]$ *be a graded ring. Assume that we have a linear map* $M: B \to \mathcal{D}(U)$ *such that, if* $q \in B$ *is homogeneous of degree* d, *then* $[M_q, L] = 0$, M_q *has homogeneity degree* $-d$, *and*

$$M_q = q(\partial_{y_1}, \dots, \partial_{y_n}) + l.o.t.$$

Then M *is a ring homomorphism and* $M_q = L_q$ *for all* $q \in \mathbf{C}[q_1, \dots, q_n]$.

Proof. 1) If there exist two different operators M_q and M'_q with these properties, take $M_q - M'_q$. This operator has degree of homogeneity $-d$, but order smaller than d. Therefore, its symbol $S(x, y)$ is not a polynomial. On the other hand, since the symbol of L is $\sum y_i^2$, we get that $[L, M_q - M'_q] = 0$ implies $\{\sum y_i^2, S(x, y)\} = 0$. Write S in the form $K(x, y)/H(x)$ with K is a polynomial, and $H(x)$ a homogeneous polynomial of positive degree t (we assume that $K(x, y)$ and $H(x)$ have no common irreducible factors). Then

$$0 = \{\sum y_i^2, S(x, y)\} = 2\,\frac{\sum_{i=1}^n y_i K_{x_i}(x, y)H(x) - \sum_{i=1}^n y_i H_{x_i}(x)K(x, y)}{H(x)^2}.$$

Since $\sum_{i=1}^n x_i H_{x_i}(x) = tH(x)$, we have $\sum_{i=1}^n y_i H_{x_i}(x)K(x, y) \neq 0$. So $H(x)$ must divide this polynomial and, by our assumptions, this implies that it must divide the polynomial $\sum_{i=1}^n y_i H_{x_i}(x)$ whose degree in x is $t - 1$. This is a contradiction.

2) Let $q, p \in B$ be two homogeneous elements. Then $M_q M_p$ and M_{pq} both satisfy the same homogeneity assumptions. Hence they are equal by 1).

Finally if $q \in \mathbf{C}[q_1, \dots, q_n]$, both M_q and L_q satisfy the same homogeneity assumptions. Hence they are equal by 1). □

The required extension to the ring of m-quasi-invariants is then provided by the following

THEOREM 2.10 ([CV1, CV2]). *Let $c = m: \Sigma \to \mathbf{Z}_+$. The following two conditions are equivalent for a homogeneous polynomial $q \in \mathbf{C}[\mathfrak{h}^*]$ of degree d.*

1) *There exists a differential operator*

$$L_q = q(\partial_{y_1}, \dots, \partial_{y_n}) + l.o.t.$$

of homogeneity degree $-d$, such that $[L_q, L] = 0$.

2) *q is an m-quasi-invariant homogeneous of degree d.*

Using this, we can extend system (5) to the system

$$L_p\psi = p(k)\psi\,, \quad p \in Q_m\,, \quad k \in \operatorname{Spec} Q_m = X_m\,. \tag{6}$$

(Recall that, as a set, $X_m = \mathfrak{h}$.) Near a generic point $x_0 \in \mathfrak{h}$, system (6) has a one dimensional space of solutions, thus there exists a unique up to scaling solution $\psi(k,x)$, which can be expressed in elementary functions. This solution is called the *Baker-Akhiezer function*, and has the form

$$\psi(k,x) = P(k,x)\, e^{(k,x)}$$

with $P(k,x)$ a polynomial of the form $\delta(x)\delta(k) + l.o.t.$ and $e^{(k,x)}$ denotes the exponential function computed in the scalar product (k,x). Furthermore, it can be shown that $\psi(k,x) = \psi(x,k)$ (see [CV1, CV2, FV]).

These results motivate the following terminology. The variety X_m is called *the spectral variety* of the Calogero-Moser system for the multiplicity function m, and Q_m is called *the spectral ring* of this system.

2.7 AN EXAMPLE

EXAMPLE 2.11. Let $W = \mathbf{Z}/2$, $\mathfrak{h} = \mathbf{C}$, $m = 1$. As we have seen, Q_m has a basis given by the monomials $\{x^{2i}\} \cup \{x^{2i+3}\}$, $i \geq 0$. Let us set for such a monomial, $L_{x^r} = L_r$, and $\partial = \frac{d}{dx}$. Then we have

$$L_0 = 1, \quad L_2 = \partial^2 - \frac{2}{x}\partial, \quad L_3 = \partial^3 - \frac{3}{x}\partial^2 + \frac{3}{x^2}\partial\,.$$

As for the others, $L_{2j} = L_2^j$, $L_{2j+3} = L_2^j L_3$. (Note that L_1 is not defined). The system (6) in this case is

$$\begin{cases} \psi'' - \dfrac{2}{x}\psi' = k^2\psi\,, \\ \psi''' - \dfrac{3}{x}\psi'' + \dfrac{3}{x^2}\psi' = k^3\psi\,. \end{cases}$$

The solution can easily be computed by differentiating the first equation and then subtracting the second, thus obtaining the new system

$$\begin{cases} \psi'' - \frac{2}{x}\psi' = k^2\psi\,, \\ \psi'' - (\frac{1}{x} + k^2 x)\psi' = -k^3 x\psi\,. \end{cases}$$

Taking the difference, we get the first order equation

$$\psi' = \frac{k^2 x}{kx-1}\psi\,,$$

whose solution (up to constants) is given by $\psi = (kx-1)\,e^{kx}$.

In fact, one can easily calculate ψ_m for a general m.

PROPOSITION 2.12. $\psi_m(k,x) = (x\partial - 2m + 1)(x\partial - 2m - 1)\cdots(x\partial - 1)\,e^{kx}$.

Proof. We could use the direct method of Example 2.11, but it is more convenient to proceed differently. Namely, we have

$$(\partial^2 - \frac{2m}{x}\partial)(x\partial - 2m + 1) = (x\partial - 2m + 1)(\partial^2 - \frac{2(m-1)}{x}\partial)$$

as it is easy to verify directly. So using induction on m starting with $m = 0$, we get

$$(\partial^2 - \frac{2m}{x}\partial)\psi_m(k,x) = (x\partial - 2m + 1)(\partial^2 - \frac{2(m-1)}{x}\partial)\psi_{m-1}(k,x) = k^2\psi_m(k,x)\,,$$

and $\psi_m(k,x)$ is our solution. □

3. LECTURE 3

3.1 SHIFT OPERATOR AND CONSTRUCTION OF THE BAKER-AKHIEZER FUNCTION

In Lecture 2, we have introduced the Baker-Akhiezer function $\psi(k,x)$ for the operator

$$L = \Delta - \sum_{s\in\Sigma} \frac{2c_s}{\alpha_s(x)}\partial_{\alpha_s}\,.$$

The way to construct $\psi(k,x)$ is via the Opdam shift operator. Given a function $m\colon \Sigma \to \mathbf{Z}_+$, Opdam showed in [Op1] that there exists a unique W-invariant

differential operator S_m of the form $\delta_m(x)\delta_m(\partial_x)+l.o.t.$, with $\delta_m(x) = \prod_{s\in\Sigma} \alpha_s^{m_s}$ such that

$$L_q S_m = S_m q(\partial)$$

for every $q \in \mathbf{C}[\mathfrak{h}] = \mathbf{C}[q_1, \dots, q_n]$. From this, if we set $\psi(k,x) = S_m e^{(k,x)}$, we get

$$L_q\psi = S_m q(\partial)\, e^{(k,x)} = q(k)\psi\,, \tag{7}$$

$q \in \mathbf{C}[q_1, \dots, q_n]$.

We claim that equation (7) must in fact hold for all $q \in Q_m$. Indeed, near a generic point x, the functions $\psi(wk,x)$ are obviously linearly independent and satisfy (7) for symmetric q. Thus, they are a basis in the space of solutions (we know that this space is $|W|$-dimensional). Consider the matrix of L_q in this basis for any $q \in Q_m$. Since $\psi(k,x)$ is a polynomial multiplied by $e^{(k,x)}$, this matrix must be diagonal with eigenvalues $q(k)$, as desired.

EXAMPLE 3.1. As we have seen in the previous section, for $W = \mathbf{Z}/2$ and $\mathfrak{h} = \mathbf{C}$,

$$S_m = (x\partial - 2m + 1)(x\partial - 2m - 1)\cdots(x\partial - 1)\,.$$

3.2 BEREST'S FORMULA FOR L_q

We are now going to give an explicit construction of the operators L_q for any $q \in Q_m$.

Let us identify, using our W-invariant scalar product, $\mathfrak{h}$ with $\mathfrak{h}^*$, and let us choose a orthonormal basis $x_1, \dots, x_n$ in $\mathfrak{h}^*$. If $x \in \mathfrak{h}^*$, we will write D_x for the Dunkl operator relative to the vector in $\mathfrak{h}$ corresponding to x under our identification. Thus

$$L = \sum_{i=1}^{n} D_{x_i}^2\,.$$

PROPOSITION 3.2 (Berest [Be]). *If $q \in Q_m$ is a homogeneous element of degree d, then*

$$(\operatorname{ad} L)^{d+1} q = 0\,.$$

Proof. It is enough to prove that

$$((\operatorname{ad} L)^{d+1} q)\psi(k,x) = 0\,.$$

Indeed, it follows from the definition of $\psi(k,x)$ that in the ring $\mathcal{D}(U)$ this implies: $((\operatorname{ad} L)^{d+1} q)\, S_m = 0$, so that $(\operatorname{ad} L)^{d+1}\, q = 0$, since $\mathcal{D}(U)$ is a domain.

Given $q \in Q_m$, we will denote by $L_q^{(k)}$ the operator $q(D_{k_1}, \ldots, D_{k_n})$. Notice that since $\psi(k,x) = \psi(x,k)$, we have $L_q^{(k)}\psi = q(x)\psi$. Thus we deduce, for $p, q, r \in Q_m$,

$$\begin{aligned} L_q r(x) L_p \psi &= L_q r(x) p(k) \psi = p(k) L_q r(x)\, \psi \\ &= p(k) L_q L_r^{(k)} \psi = p(k) L_r^{(k)} L_q \psi = p(k) L_r^{(k)} q(k)\, \psi\,. \end{aligned}$$

It follows that

$$(\operatorname{ad} L)^{d+1} q\psi = (-1)^{d+1} (\operatorname{ad}(\sum_{i=1}^{n} k_i^2))^{d+1} L_q^{(k)} \psi\,.$$

Since L_q is a differential operator of degree d, we get $\operatorname{ad}(\sum_{i=1}^n k_i^2)^{d+1} L_q^{(k)} = 0$, as desired. □

Notice now that the operator $(adL)^d q(x)$ commutes with L. Its symbol is given by $(\operatorname{ad}\Delta)^d q(x) = 2^d d! q(\partial)$. So we deduce the following

COROLLARY 3.3 (Berest's formula, [Be]). *If $q \in Q_m$ is homogeneous of degree d, then*

$$L_q = \frac{1}{2^d d!} (\operatorname{ad} L)^d q(x)\,.$$

Proof. This is clear from Proposition 2.8, once we remark that $(\operatorname{ad} L)^d q(x)$ has the required homogeneity. □

We want to give a representation theoretical interpretation of what we have just seen. Consider the three operators

$$F = \frac{\sum_{i=1}^n x_i^2}{2}\,, \quad E = -\frac{L}{2}\,, \quad H = [E,F]\,. \tag{8}$$

It is easy to check that $[H,E] = 2E$, $[H,F] = -2F$. We deduce that the elements E, F, H span an $\mathfrak{sl}(2)$ Lie subalgebra of $\mathcal{D}(U)$. Thus $\mathfrak{sl}(2)$ acts by conjugation on $\mathcal{D}(U)$. We can then reformulate Proposition 3.2 as follows:

PROPOSITION 3.4. *Any polynomial $q \in Q_m$ of degree d is a lowest weight vector for the $\mathfrak{sl}(2)$-action of weight $-d$ and generates a finite dimensional module (necessarily of dimension $d+1$) for which L_q is a highest weight vector.*

Proof. An easy direct computation shows that

$$H = [E, F] = -\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} + C,$$

where C is a constant. Thus if q is homogeneous of degree d, we have $[H, L_q] = dL_q$.

This and the fact that $[L, L_q] = 0$, implies that L_q is a highest weight vector of weight d. Also since F is a polynomial, we deduce that $\operatorname{ad} F^{d+1} L_q = 0$, so that L_q generates a $(d+1)$-dimensional irreducible $\mathfrak{sl}(2)$-module. □

One last property about these operators is given by

PROPOSITION 3.5 ([FV]). *For any $q \in Q_m$, the operator L_q preserves Q_m.*

Proof. Let us begin by proving that L preserves Q_m.

Take $f \in Q_m$, so that for any $s \in \Sigma$, $f - {}^s f = \alpha_s^{2m_s+1} t$, $t \in \mathbf{C}[\mathfrak{h}]$. Let us start by showing that Lf is a polynomial. Clearly $Lf = \delta_*^{-1} q$, with $q \in \mathbf{C}[\mathfrak{h}]$, and $\delta_* = \prod_{s:m_s \neq 0} \alpha_s$. Since L is W-invariant, $Lf - {}^s(Lf) = L(f - {}^s f)$ is clearly divisible by $\alpha_s^{2m_s-1}$ if $m_s > 0$. In particular, it is always regular along the reflection hyperplane of s. On the other hand, since $Lf - {}^s(Lf) = \delta_*^{-1}(q + {}^s q)$, we deduce that $q + {}^s q$ is divisible by α_s if $m_s > 0$. But then $q = ((q + {}^s q) + (q - {}^s q))/2$ is divisible by α_s if $m_s > 0$, hence it is divisible by δ_*, so that Lf lies in $\mathbf{C}[\mathfrak{h}]$.

We have already remarked that $L(f - {}^s f)$ is divisible by $\alpha_s^{2m_s-1}$ if $m_s > 0$. In fact

$$L(f - {}^s f) = (L\alpha_s^{2m_s+1}) t + \alpha_s^{2m_s} \tilde{t},$$

where $\tilde{t}$ is a suitable polynomial.

But since

$$L\alpha_s^{2m_s+1} = 2m_s(2m_s+1)(\alpha_s, \alpha_s)\alpha_s^{2m_s-1} - 2m_{s'}(2m_s+1) \sum_{s' \in \Sigma} (\alpha_{s'}, \alpha_s) \frac{\alpha_s^{2m_s}}{\alpha_{s'}}$$

$$= -2m_{s'}(2m_s+1) \sum_{s' \in \Sigma, s' \neq s} (\alpha_{s'}, \alpha_s) \frac{\alpha_s^{2m}}{\alpha_{s'}},$$

we deduce that $L(f - {}^s f)$ is divisible by $\alpha_s^{2m_s}$. On the other hand, since $L(f - {}^s f) = Lf - {}^s(Lf)$, this polynomial is either zero or it must vanish to odd order on the reflection hyperplane of s. We deduce that it must be divisible by $\alpha_s^{2m_s+1}$, proving that $Lf \in Q_m$.

We now pass to a general L_q, $q \in Q_m$. We may assume that q is homogeneous of, say, degree d. By Corollary 3.3 we have that L_q is a non zero multiple of $(adL)^d(q)$. Since both q and L preserve Q_m, our claim follows. □

3.3 Differential operators on X_m

Now let us return to the algebra of differential operators $\mathcal{D}(X_m)$. Notice that $\mathcal{D}(X_m)$ contains two commutative subalgebras (both isomorphic to Q_m). The first is Q_m itself, the second is the subalgebra $Q_m^\dagger$ consisting of the differential operators of the form L_q with $q \in Q_m$. It is possible to prove

THEOREM 3.6 ([BEG]). *$\mathcal{D}(X_m)$ is generated by Q_m and $Q_m^\dagger$.*

Notice that by Corollary 3.3 we in fact have that $\mathcal{D}(X_m)$ is generated by Q_m and by L.

EXAMPLE 3.7. If $W = \mathbf{Z}/2$, $\mathfrak{h} = \mathbf{C}$ we get that $\mathcal{D}(X_m)$ is generated by the operators

$$x^2, \quad x^{2m+1}, \quad \frac{d^2}{dx^2} - \frac{2m}{x}\frac{d}{dx}.$$

Theorem 3.6 together with Proposition 3.4, imply

COROLLARY 3.8 ([BEG]). *$\mathcal{D}(X_m)$ is locally finite dimensional under the action of the Lie algebra $\mathfrak{sl}(2)$ defined in (8).*

This Corollary implies that our $\mathfrak{sl}(2)$ action on $\mathcal{D}(X_m)$ can be integrated to an action of the group $SL(2)$. In particular we have

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} q = L_q$$

for all $q \in Q_m$. This transformation is a generalization of the Fourier transform, since it reduces to the usual Fourier transform on differential operators on $\mathfrak{h}$ when $m = 0$.

EXAMPLE 3.9. If $W = \mathbf{Z}/2$, $\mathfrak{h} = \mathbf{C}$, we get that the monomials $\{x^{2i}\} \cup \{x^{2i+2m+1}\}$ are (up to constants) all lowest weight vectors for the $\mathfrak{sl}(2)$ action on $\mathcal{D}(X_m)$. x^n has weight $-n$. We deduce that $\mathcal{D}(X_m)$ is isomorphic as a $\mathfrak{sl}(2)$-module to the direct sum of the irreducible representations of dimension $n+1$ for n even or $n = 2(m+i)+1$, each with multiplicity one.

3.4 THE CHEREDNIK ALGEBRA

Let us now return to the algebra $\mathcal{A}$ of operators on U generated by $\mathcal{D}(U)$ and W. This algebra contains the Dunkl operators

$$D_y := \partial_y + \sum_{s\in\Sigma} c_s \frac{(\alpha_s, y)}{\alpha_s}(s-1).$$

LEMMA 3.10. *The following relations hold:*

$$[x_i, x_j] = [D_{x_i}, D_{x_j}] = 0, \quad \forall 1 \le i,j \le n$$

$$[D_{x_i}, x_j] = \delta_{i,j} + \sum_{s\in\Sigma} c_s \frac{(x_i, \alpha_s)(x_j, \alpha_s)}{(\alpha_s, \alpha_s)} s, \quad \forall 1 \le i,j \le n$$

$$wxw^{-1} = w(x), \quad wD_y w^{-1} = D_{w(y)}, \quad \forall w \in W,\ x \in \mathfrak{h}^*,\ y \in \mathfrak{h}.$$

Proof. The proof is an easy computation, except for the relations $[D_{x_i}, D_{x_j}] = 0$, which follow from Theorem 2.6. □

This lemma motivates the following definition.

DEFINITION 3.11 (see e.g. [EG]). The *Cherednik algebra* H_c is an associative algebra with generators $x_i, y_i, i = 1, \dots, n$, and $w \in W$, with defining relations

$$[x_i, x_j] = [y_i, y_j] = 0, \quad \forall 1 \le i,j \le n$$

$$[y_i, x_j] = \delta_{i,j} + \sum_{s\in\Sigma} c_s \frac{(x_i, \alpha_s)(x_j, \alpha_s)}{(\alpha_s, \alpha_s)} s, \quad \forall 1 \le i,j \le n$$

$$wxw^{-1} = w(x),\ wyw^{-1} = w(y),\ w \cdot w' = ww',\ \forall w, w' \in W,\ x \in \mathfrak{h}^*,\ y \in \mathfrak{h}.$$

This algebra was introduced by Cherednik as a rational limit of his double affine Hecke algebra defined in [Ch]. Notice that if $c = 0$ then $H_0 = \mathcal{D}(\mathfrak{h}) \rtimes \mathbf{C}[W]$.

Lemma 3.10 implies that the algebra H_c is equipped with a homomorphism $\phi\colon H_c \to \mathcal{A}$, given by $w \to w$, $x_i \to x_i$, $y_i \to D_{x_i}$.

Cherednik proved the following theorem.

THEOREM 3.12 (Poincaré-Birkhoff-Witt theorem). *The multiplication map*

$$\mu\colon \mathbf{C}[\mathfrak{h}] \otimes \mathbf{C}[\mathfrak{h}^*] \otimes \mathbf{C}[W] \to H_c$$

given by $\mu(f(x) \otimes g(y) \otimes w) = f(x)\, g(y) w$ *is an isomorphism of vector spaces.*

Proof. It is easy to see that the map μ is surjective. Thus, we only have to show that it is injective. In other words, we need to show that monomials $x_1^{i_1} \dots x_n^{i_n} y_1^{j_1} \dots y_n^{j_n} w$ are linearly independent in H_c. To do this, it suffices to show that the images of these monomials under the homomorphism ϕ, i.e. $x_1^{i_1} \dots x_n^{i_n} D_{x_1}^{j_1} \dots D_{x_n}^{j_n} w$, are linearly independent.

Given an element $A \in \mathcal{A}$, writing $A = \sum_{w \in W} P_w w$ with $P_w \in \mathcal{D}(U)$ we define the order of A, ord$A$, as the maximum of the orders of the P_w's. Notice that ord$AB \leq$ ord$A$+ord$B$. We now remark that for any sequence of non negative indices $(i_1, \dots, i_n)$,

$$D_{x_1}^{i_1} \cdots D_{x_n}^{i_n} = \partial_{x_1}^{i_1} \cdots \partial_{x_n}^{i_n} + l.o.t.$$

Indeed this is true for D_{x_i}. We proceed by induction on $r = i_1 + \cdots + i_n$. We can clearly assume $i_1 > 0$, so by induction,

$$D_{x_1}^{i_1} \cdots D_{x_n}^{i_n} = (\partial_{x_1} + l.o.t.)(\partial_{x_1}^{i_1 - 1} \cdots \partial_{x_n}^{i_n} + l.o.t.) = \partial_{x_1}^{i_1} \cdots \partial_{x_n}^{i_n} + l.o.t.$$

From this we deduce that for any pair of multiindices $I = (i_1, \dots, i_n)$, $J = (j_1, \dots, j_n)$, $w \in W$, setting $x_I = {x_1}^{i_1} \cdots {x_n}^{i_n}$, $D_J = D_{x_1}^{j_1} \cdots D_{x_n}^{j_n}$, $\partial_J = \partial_{x_1}^{j_1} \cdots \partial_{x_n}^{j_n}$, we have

$$x_I D_J w = x_I \partial_J w + l.o.t.$$

Using this and the linear independence of the elements $x_I \partial_J w$, it is immediate to conclude that the elements $x_I D_J w$ are linearly independent, proving our claim. □

REMARK 1. We see that the homomorphism ϕ identifies H_c with the subalgebra of $\mathcal{A}$ generated by $\mathbf{C}[\mathfrak{h}]$, the Dunkl operators D_y, $y \in \mathfrak{h}$ and W.

REMARK 2. Another way to state the PBW theorem is the following. Let $F^\bullet$ be a filtration on H_c defined by $\deg(x_i) = \deg(y_i) = 1$, $\deg(w) = 0$. Then we have a natural surjective mapping from $\mathbf{C}[\mathfrak{h} \times \mathfrak{h}^*] \rtimes W$ to the associated graded algebra $\mathrm{gr}(H_c)$. The PBW theorem claims that this map is in fact an isomorphism.

3.5 THE SPHERICAL SUBALGEBRA

Let us now introduce the idempotent

$$e = \frac{1}{W} \sum_{w \in W} w \in \mathbf{C}[W].$$

DEFINITION 3.13. The *spherical subalgebra* of H_c is the algebra eH_ce.

Notice that $1 \notin eH_ce$. On the other hand, since $ex = xe = e$ for $x \in eH_ce$, e is the unit for the spherical subalgebra. We can embed both $\mathbf{C}[\mathfrak{h}^*]^W$ and $\mathbf{C}[\mathfrak{h}]^W$ in the spherical subalgebra as follows. Take $f \in \mathbf{C}[\mathfrak{h}^*]^W$ (the other case is identical) and set $m_e(f) = fe$. Since f is invariant, we have $efe = fe^2 = fe = m_e(f)$, so that m_e actually maps $\mathbf{C}[\mathfrak{h}^*]^W$ to eH_ce. The injectivity is clear from the PBW-theorem. As for the fact that m_e is a homomorphism, we have $m_e(fg) = fge = fge^2 = fege = m_e(f)m_e(g)$. From now on, we will consider both $\mathbf{C}[\mathfrak{h}^*]^W$ and $\mathbf{C}[\mathfrak{h}]^W$ as subalgebras of the spherical subalgebra.

3.6 CATEGORY $\mathcal{O}$

We are now going to study representations of the algebras H_c and eH_ce.

DEFINITION 3.14. The category $\mathcal{O}(H_c)$ (resp. $\mathcal{O}(eH_ce)$) is the full subcategory of the category of H_c-modules (resp. eH_ce-modules) whose objects are the modules M such that

1) M is finitely generated.

2) For all $v \in M$, the subspace $\mathbf{C}[\mathfrak{h}^*]^W v \subset M$ is finite dimensional.

We can define a functor

$$F\colon \mathcal{O}(H_c) \to \mathcal{O}(eH_ce)$$

by setting $F(M) = eM$. It is easy to show that $F(M)$ is an object of $\mathcal{O}(eH_ce)$.

We are now going to explain how to construct some modules in $\mathcal{O}(H_c)$ which, by analogy with the case of enveloping algebras of semisimple Lie algebras, we will call Whittaker and Verma modules. First, take $\lambda \in \mathfrak{h}^*$. Denote by $W_\lambda \subset W$ the stabilizer of λ. Take an irreducible W_λ-module τ. We define a structure of $\mathbf{C}[\mathfrak{h}^*] \rtimes \mathbf{C}[W_\lambda]$-module on τ by

$$(fw)v = f(\lambda)(wv) \quad \forall v \in \tau,\ w \in W_\lambda,\ f \in \mathbf{C}[\mathfrak{h}^*].$$

It is easy to see that this action is well defined and we denote this module by $\lambda\#\tau$. We can then consider the H_c-module

$$M(\lambda, \tau) = H_c \otimes_{\mathbf{C}[\mathfrak{h}^*] \rtimes \mathbf{C}[W_\lambda]} \lambda\#\tau\,.$$

This is called a Whittaker module. In the special case $\lambda = 0$ (and hence $W_\lambda = W$), the module $M(0, \tau)$ is called a Verma module. It is clear that these are objects of $\mathcal{O}$. Notice that as $\mathbf{C}[\mathfrak{h}] \rtimes \mathbf{C}[W]$-module, $M(\lambda, \tau) = \mathbf{C}[\mathfrak{h}] \otimes_{\mathbf{C}} \mathbf{C}[W] \otimes_{\mathbf{C}[W_\lambda]} \tau$.

EXAMPLE 3.15. If $\lambda = 0$ and $\tau = \mathbf{1}$ is the trivial representation of W, the Verma module $M(0, \mathbf{1}) = \mathbf{C}[\mathfrak{h}]$. The action of $\mathbf{C}[\mathfrak{h}]$ is given by multiplication, that of $\mathbf{C}[\mathfrak{h}^*]$ is generated by the Dunkl operators and W acts in the usual way.

3.7 GENERIC c

Opdam and Rouquier have recently studied the structure of the categories $\mathcal{O}(H_c)$, $\mathcal{O}(eH_ce)$, and found that it is especially simple if c is "generic" in a certain sense. Namely, recall that for a W-invariant function $q\colon \Sigma \to \mathbf{C}^*$ one can define the *Hecke algebra* $\mathrm{He}_q(W)$ to be the quotient of the group algebra of the fundamental group of U/W by the relations $(T_s - 1)(T_s + q_s) = 0$, where T_s is the image in U/W of a small half-circle around the hyperplane of s in the counterclockwise direction. It is well known that $\mathrm{He}_q(W)$ is an algebra of dimension $|W|$, which coincides with $\mathbf{C}[W]$ if $q = 1$. It is also known that $\mathrm{He}_q(W)$ is semisimple (and isomorphic to $\mathbf{C}[W]$ as an algebra) unless q_s belongs for some s to a finite set of roots of unity depending on W (see [Hu]).

DEFINITION 3.16. The function c is said to be *generic* if for $q = e^{2\pi ic}$, the Hecke algebra $\mathrm{He}_q(W)$ is semisimple.

In particular, any irrational c is generic, and (more important for us) an integer valued c is generic (since in this case $q = 1$). We can now state the following central result:

THEOREM 3.17 (Opdam-Rouquier [OR]; see also [BEG] for an exposition). *If c is generic (in particular, if c takes non negative integer values), then the irreducible objects in $\mathcal{O}$ are exactly the modules $M(\lambda, \tau)$. Moreover, the category $\mathcal{O}$ is semisimple.*

We also have

THEOREM 3.18 ([OR]). *If c is generic then the functor F is an equivalence of categories.*

From Theorem 3.17 we can deduce

THEOREM 3.19 ([BEG]). *If c is generic, then H_c is a simple algebra.*

In the case $c = 0$, we get the simplicity of $\mathbf{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \rtimes \mathbf{C}[W]$, which is well known.

3.8 THE LEVASSEUR-STAFFORD THEOREM AND ITS GENERALIZATION

Let us now recall a result of Levasseur and Stafford:

THEOREM 3.20 ([LS]). *If G is a finite group acting on a finite dimensional vector space V over the complex numbers, then the ring $\mathcal{D}(V)^G$ is generated by the subrings $\mathbf{C}[V]^G$ and $\mathbf{C}[V^*]^G$.*

As an example, notice that if we let $\mathbf{Z}/n\mathbf{Z}$ act on the complex line by multiplication by the n^{th} roots of 1, we deduce that the operator $x\frac{d}{dx}$ can be expressed as a non commutative polynomial in the operators x^n and $\frac{d^n}{dx^n}$, a non-obvious fact. We note also that this theorem has a purely "quantum" nature, i.e. the corresponding "classical" statement, saying that the Poisson algebra $\mathbf{C}[V\times V^*]^G$ is generated, as a Poisson algebra, by $\mathbf{C}[V]^G$ and $\mathbf{C}[V^*]^G$, is in fact false, already for $V=\mathbf{C}$ and $G=\mathbf{Z}/n\mathbf{Z}$.

One can prove a similar result for the algebra eH_ce. Namely, recall that the algebra eH_ce contains the subalgebras $\mathbf{C}[\mathfrak{h}]^W$, and $\mathbf{C}[\mathfrak{h}^*]^W$.

THEOREM 3.21 ([BEG]). *If c is generic then the two subalgebras $\mathbf{C}[\mathfrak{h}]^W$ and $\mathbf{C}[\mathfrak{h}^*]^W$ generate eH_ce.*

Notice that if $c=0$, then $eH_0e=\mathcal{D}(\mathfrak{h})^W$, so Theorem 3.21 reduces to the Levasseur-Stafford theorem.

REMARK. It is believed that this result holds without the assumption of generic c. Moreover, it is known to be true for all c if W is a Weyl group not of type E and F, since in this case Wallach proved that the corresponding classical statement for Poisson algebras holds true. Nevertheless, the genericity assumption is needed for the proof, because, similarly to the proof of the Levasseur-Stafford theorem, it is based on the simplicity of H_c.

3.9 THE ACTION OF THE CHEREDNIK ALGEBRA TO QUASI-INVARIANTS

We now go back to the study of Q_m. Notice that the algebra eH_me acts on $\mathbf{C}[\mathfrak{h}]^W$, since e gives the W-equivariant projection of $\mathbf{C}[\mathfrak{h}]$ onto $\mathbf{C}[\mathfrak{h}]^W$. It is clear that this action is by differential operators. For instance, the subalgebra $\mathbf{C}[\mathfrak{h}]^W\subset eH_me$ acts by multiplication. Also, an element $q\in\mathbf{C}[\mathfrak{h}^*]^W\subset eH_me$ acts via the operator $q(D_{x_1},\dots,D_{x_n})$. By definition this operator coincides with L_q on $\mathbf{C}[\mathfrak{h}]^W$.

The following important theorem shows that this action extends to Q_m.

THEOREM 3.22 ([BEG]). *There exists a unique representation of the algebra eH_me on Q_m in which an element $q \in \mathbf{C}[\mathfrak{h}]^W$ acts by multiplication and an element $q \in \mathbf{C}[\mathfrak{h}^*]^W$ by L_q.*

Proof. Since by Proposition 3.5, L_q preserves Q_m, we get a uniquely defined representation of the subalgebra of eH_me generated by $\mathbf{C}[\mathfrak{h}]^W$ and $\mathbf{C}[\mathfrak{h}^*]^W$ on Q_m. The result now follows from Theorem 3.21. □

3.10 PROOF OF THEOREM 1.8

Finally we can prove Theorem 1.8.

To do this, observe that as an eH_me-module, Q_m is in the category $\mathcal{O}(eH_me)$, and $\mathbf{C}[\mathfrak{h}^*]^W$ acts locally nilpotently in Q_m (by degree arguments). We can now apply Theorem 3.18 and Theorem 3.17 and deduce that Q_m is a direct sum of modules of the form $eM(0,\tau)$. As a $\mathbf{C}[\mathfrak{h}] \rtimes \mathbf{C}[W]$-module, $M(0,\tau) = \mathbf{C}[\mathfrak{h}] \otimes \tau$. On the other hand, by Chevalley's theorem, there is an isomorphism $\mathbf{C}[\mathfrak{h}] \simeq \mathbf{C}[\mathfrak{h}]^W \otimes \mathbf{C}[W]$, commuting with the action of W and $\mathbf{C}[\mathfrak{h}]^W$. Thus we get an isomorphisms of $\mathbf{C}[\mathfrak{h}]^W$-modules

$$eM(0,\tau) \simeq (M(0,\tau))^W \simeq \mathbf{C}[\mathfrak{h}]^W \otimes (\mathbf{C}[W] \otimes \tau)^W \simeq \mathbf{C}[\mathfrak{h}]^W \otimes \tau,$$

proving that $eM(0,\tau)$ and hence Q_m is a free $\mathbf{C}[\mathfrak{h}]^W$-module. □

EXAMPLE 3.23. For $W = \mathbf{Z}/2$ and $\mathfrak{h} = \mathbf{C}$, take the polynomials 1, x^{2m+1}. Notice that $L(1) = L(x^{2m+1}) = 0$ while $s(1) = 1$, $s(x^{2m+1}) = -x^{2m+1}$, $s \in \mathbf{Z}/2$ being the element of order two. It follows that Q_m as a eH_me-module is the direct sum of $\mathbf{C}[x^2] \oplus x^{2m+1}\mathbf{C}[x^2]$. These modules are irreducible. Moreover, $\mathbf{C}[x^2] \simeq eM(0,\mathbf{1})$, $x^{2m+1}\mathbf{C}[x^2] \simeq eM(0,\varepsilon)$, ε being the sign representation.

3.11 PROOF OF THEOREM 1.15

Let I be a nonzero two-sided ideal in $\mathcal{D}(X_m)$. First we claim that I nontrivially intersects Q_m. Indeed, otherwise let $K \in I$ be a lowest order nonzero element in I. Since the order of K is positive, there exists $f \in Q_m$ such that $[K,f] \neq 0$. Then $[K,f] \in I$ is of smaller order than K, a contradiction.

Now let $f \in Q_m$ be an element of I. Then $g = \prod_{w \in W} {}^w f \in I$. But g is W-invariant. This shows that the intersection J of I with the subalgebra H_m in $\mathcal{D}(X_m)$ is nonzero. But H_m is simple by Theorem 3.19, so $J = H_m$. Hence, $1 \in J \subset I$, and $I = \mathcal{D}(X_m)$. □

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