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cohomology class in $H^1(M,\underline{\mathrm{U}(1)}) = H^2(M,\mathbf{Z})$ defined by this cocycle is the Chern class of the line bundle. Chatterjee-Hitchin [10, 18, 17] suggested to realize classes in $H^3(M,\mathbf{Z})$ in a similar fashion, replacing U(1)-valued functions with Hermitian line bundles. They define a gerbe to be a collection of Hermitian transition line bundles $L_{ab} \to U_a \cap U_b$ and a trivialization, i.e. unit length section, t_{abc} of the line bundle $(\delta L)_{abc} = L_{bc}L_{ac}^{-1}L_{ab}$ over triple intersections. These trivializations have to satisfy a compatibility relation over quadruple intersections,

$$(\delta t)_{abcd} \equiv t_{bcd}t_{acd}^{-1}t_{abd}t_{abc}^{-1} = 1\,,$$

which makes sense since $(\delta t)_{abcd}$ is a section of the *canonically* trivial bundle. (Each factor L_{ab} cancels with a factor L_{ab}^{-1}.) After passing to a refinement of the cover, such that all L_{ab} become trivializable, and picking trivializations, t_{abc} is simply a Čech cocycle of degree 2, hence defines a class in $H^2(M,\underline{\mathrm{U}(1)}) = H^3(M,\mathbf{Z})$. The class is independent of the choices made in this construction, and is called the *Dixmier-Douady class* of the gerbe.

Note that in practice, it is often not desirable to pass to a refinement. For example, if M is a connected, oriented 3-manifold, the generator of $H^3(M,\mathbf{Z}) = \mathbf{Z}$ can be described in terms of the cover U_1, U_2, where U_1 is an open ball around a given point $p \in M$, and $U_2 = M\backslash\{p\}$, using the degree one line bundle over $U_1 \cap U_2 \cong S^2 \times (0,1)$.

2.2 Bundle gerbes

Bundle gerbes were invented by Murray [24], generalizing the following construction of line bundles. Let $\pi\colon X \to M$ be a fiber bundle, or more generally a surjective submersion. (Different components of X may have different dimensions.) For each $k \geq 0$ let $X^{[k]}$ denote the k-fold fiber product of X with itself. There are $k+1$ projections $\partial^i\colon X^{[k+1]} \to X^{[k]}$, omitting the ith factor in the fiber product. Suppose we are given a smooth function $\chi\colon X^{[2]} \to \mathrm{U}(1)$, satisfying a cocycle condition $\delta\chi = 1$ where

$$\delta\chi := \partial_0^*\chi\partial_1^*\chi^{-1}\partial_2^*\chi\colon X^{[3]} \to \mathrm{U}(1)\,.$$

Then χ determines a Hermitian line bundle $L \to M$, with fibers at $m \in M$ the space of all linear maps $\phi\colon X_m = \pi^{-1}(m) \to \mathbf{C}$ such that $\phi(x) = \chi(x,x')\phi(x')$. Given local sections $\sigma_a\colon U_a \to X$ of X, the pull-backs of χ under the maps $(\sigma_a,\sigma_b)\colon U_a \cap U_b \to X^{[2]}$ give transition functions χ_{ab} for the line bundle.

Again, replacing U(1)-valued functions by line bundles in this construction, one obtains a model for gerbes: A bundle gerbe is given by a line bundle $L \to X^{[2]}$ and a trivializing section t of the line bundle $\delta L = \partial_0^*L\otimes\partial_1^*L^{-1}\otimes\partial_2^*L$

over $X^{[3]}$, satisfying a compatibility condition $\delta t = 1$ over $X^{[4]}$ (which makes sense since δt is a section of the canonically trivial bundle $\delta\delta L$). Given local sections $\sigma_a \colon U_a \to X$, one can pull these data back under the maps $(\sigma_a, \sigma_b) \colon U_a \cap U_b \to X^{[2]}$ and $(\sigma_a, \sigma_b, \sigma_c) \colon U_a \cap U_b \cap U_c \to X^{[3]}$ to obtain a Chatterjee-Hitchin gerbe. The Dixmier-Douady class of (X, L, t) is by definition the Dixmier-Douady class of this Chatterjee-Hitchin gerbe; again this is independent of all choices. The Dixmier-Douady class behaves naturally under tensor product, pull-back and duals.

Notice that Chatterjee-Hitchin gerbes may be viewed as a special case of bundle gerbes, with X the disjoint union of the sets U_a in the given cover.

REMARK 2.1. In his original paper [24] Murray considered bundle gerbes only for fiber bundles, but this was found too restrictive. In [25], [29] the weaker condition (called 'locally split') is used that every point $x \in M$ admits an open neighborhood U and a map $\sigma \colon U \to X$ such that $\pi \circ \sigma = \mathrm{id}$. However, this condition seems insufficient in the smooth category, as the fiber product $X \times_M X$ need not be a manifold unless π is a submersion.

2.3 SIMPLICIAL GERBES

Murray's construction fits naturally into a wider context of *simplicial gerbes*. We refer to Mostow-Perchik's notes of lectures by R. Bott [23] and to Dupont's paper [12] for a nice introduction to simplicial manifolds, and to Stevenson [29] for their appearance in the gerbe context.

Recall that a *simplicial manifold* $M_\bullet$ is a sequence of manifolds $(M_n)_{n=0}^{\infty}$, together with *face maps* $\partial_i \colon M_n \to M_{n-1}$ for $i = 0, \dots, n$ satisfying relations $\partial_i \circ \partial_j = \partial_{j-1} \circ \partial_i$ for $i < j$. (The standard definition also involves *degeneracy maps* but these need not concern us here.) The *(fat) geometric realization* of $M_\bullet$ is the topological space $||M|| = \coprod_{n=1}^{\infty} \Delta^n \times M_n / \sim$, where Δ^n is the n-simplex and the relation is $(t, \partial_i(x)) \sim (\partial^i(t), x)$, for $\partial^i \colon \Delta^{n-1} \to \Delta^n$ the inclusion as the ith face. A (smooth) simplicial map between simplicial manifolds $M_\bullet, M'_\bullet$ is a collection of smooth maps $f_n \colon M_n \to M'_n$ intertwining the face maps; such a map induces a map between the geometric realizations.

EXAMPLES 2.2.

(a) If S is any manifold, one can define a simplicial manifold $E_\bullet S$ where $E_n S$ is the $n+1$-fold cartesian product of S, and ∂_j omits the jth factor. It is known [23] that the geometric realization $||ES||$ of this simplicial manifold is contractible. More generally, if $X \to M$ is a fiber bundle with fiber S,