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The  $L_{ij}$ , together with these isomorphisms, define a gerbe over  $SU(d+1)$ , representing the generator of  $H^3(SU(d+1), \mathbf{Z})$ .

More generally, consider any compact, simply connected, simple Lie group  $G$  of rank  $d$ . Up to conjugacy,  $G$  contains exactly  $d+1$  elements with semi-simple centralizer. (For  $G = SU(d+1)$ , these are the central elements.) Let  $\mathcal{C}_1, \dots, \mathcal{C}_{d+1} \subset G$  be their conjugacy classes. We will define an invariant open cover  $V_1, \dots, V_{d+1}$  of  $G$ , with the property that each member of this cover admits an equivariant retraction onto the conjugacy class  $\mathcal{C}_j \subset V_j$ . It turns out that every semi-simple centralizer has a distinguished central extension by  $U(1)$ . This central extension defines an equivariant bundle gerbe on  $\mathcal{C}_j$ , hence (by pull-back) an equivariant bundle gerbe over  $V_j$ . We will find that these gerbes over  $V_j$  glue together to produce a gerbe over  $G$ , using a gluing rule developed in this paper.

The organization of the paper is as follows. In Section 2 we review the theory of gerbes and pseudo-line bundles with connections, and discuss 'strong equivariance' under a group action. Section 4 describes gluing rules for bundle gerbes. Section 3 summarizes some facts about gerbes coming from central extensions. In Section 5 we give the construction of the basic gerbe over  $G$  outlined above, and in Section 6 we study the 'pre-quantization of conjugacy classes'.

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## 2. GERBES WITH CONNECTIONS

In this section we review gerbes on manifolds, along the lines of Chatterjee-Hitchin and Murray.

### 2.1 CHATTERJEE-HITCHIN GERBES

Let  $M$  be a manifold. Any Hermitian line bundle over  $M$  can be described by an open cover  $U_a$ , and transition functions  $\chi_{ab}: U_a \cap U_b \rightarrow U(1)$  satisfying a cocycle condition  $(\delta\chi)_{abc} = \chi_{bc}\chi_{ac}^{-1}\chi_{ab} = 1$  on triple intersections. The

cohomology class in  $H^1(M, \underline{U(1)}) = H^2(M, \mathbf{Z})$  defined by this cocycle is the Chern class of the line bundle. Chatterjee-Hitchin [10, 18, 17] suggested to realize classes in  $H^3(M, \mathbf{Z})$  in a similar fashion, replacing  $U(1)$ -valued functions with Hermitian line bundles. They define a gerbe to be a collection of Hermitian transition line bundles  $L_{ab} \rightarrow U_a \cap U_b$  and a trivialization, i.e. unit length section,  $t_{abc}$  of the line bundle  $(\delta L)_{abc} = L_{bc}L_{ac}^{-1}L_{ab}$  over triple intersections. These trivializations have to satisfy a compatibility relation over quadruple intersections,

$$(\delta t)_{abcd} \equiv t_{bcd}t_{acd}^{-1}t_{abd}t_{abc}^{-1} = 1,$$

which makes sense since  $(\delta t)_{abcd}$  is a section of the *canonically* trivial bundle. (Each factor  $L_{ab}$  cancels with a factor  $L_{ab}^{-1}$ .) After passing to a refinement of the cover, such that all  $L_{ab}$  become trivializable, and picking trivializations,  $t_{abc}$  is simply a Čech cocycle of degree 2, hence defines a class in  $H^2(M, \underline{U(1)}) = H^3(M, \mathbf{Z})$ . The class is independent of the choices made in this construction, and is called the *Dixmier-Douady class* of the gerbe.

Note that in practice, it is often not desirable to pass to a refinement. For example, if  $M$  is a connected, oriented 3-manifold, the generator of  $H^3(M, \mathbf{Z}) = \mathbf{Z}$  can be described in terms of the cover  $U_1, U_2$ , where  $U_1$  is an open ball around a given point  $p \in M$ , and  $U_2 = M \setminus \{p\}$ , using the degree one line bundle over  $U_1 \cap U_2 \cong S^2 \times (0, 1)$ .

## 2.2 BUNDLE GERBES

Bundle gerbes were invented by Murray [24], generalizing the following construction of line bundles. Let  $\pi: X \rightarrow M$  be a fiber bundle, or more generally a surjective submersion. (Different components of  $X$  may have different dimensions.) For each  $k \geq 0$  let  $X^{[k]}$  denote the  $k$ -fold fiber product of  $X$  with itself. There are  $k + 1$  projections  $\partial^i: X^{[k+1]} \rightarrow X^{[k]}$ , omitting the  $i$ th factor in the fiber product. Suppose we are given a smooth function  $\chi: X^{[2]} \rightarrow U(1)$ , satisfying a cocycle condition  $\delta\chi = 1$  where

$$\delta\chi := \partial_0^*\chi\partial_1^*\chi^{-1}\partial_2^*\chi: X^{[3]} \rightarrow U(1).$$

Then  $\chi$  determines a Hermitian line bundle  $L \rightarrow M$ , with fibers at  $m \in M$  the space of all linear maps  $\phi: X_m = \pi^{-1}(m) \rightarrow \mathbf{C}$  such that  $\phi(x) = \chi(x, x')\phi(x')$ . Given local sections  $\sigma_a: U_a \rightarrow X$  of  $X$ , the pull-backs of  $\chi$  under the maps  $(\sigma_a, \sigma_b): U_a \cap U_b \rightarrow X^{[2]}$  give transition functions  $\chi_{ab}$  for the line bundle.

Again, replacing  $U(1)$ -valued functions by line bundles in this construction, one obtains a model for gerbes: A bundle gerbe is given by a line bundle  $L \rightarrow X^{[2]}$  and a trivializing section  $t$  of the line bundle  $\delta L = \partial_0^*L \otimes \partial_1^*L^{-1} \otimes \partial_2^*L$