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THE BASIC GERBE OVER A COMPACT SIMPLE LIE GROUP

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ABSTRACT. Let G be a compact, simply connected simple Lie group. We give a construction of an equivariant gerbe with connection on G, with equivariant 3-curvature representing a generator of $H^3_G(G, \mathbf{Z})$. Among the technical tools developed in this context is a gluing construction for equivariant bundle gerbes.

1. INTRODUCTION

Let G be a compact, simply connected simple Lie group, acting on itself by conjugation. It is well-known that the cohomology of G, and also its equivariant cohomology, is trivial in degree less than three and that $H^3(G, \mathbf{Z})$ and $H^3_G(G, \mathbf{Z})$ are canonically isomorphic to $\mathbf{Z}$. The generator of $H^3(G, \mathbf{Z})$ is represented by a unique bi-invariant differential form $\eta \in \Omega^3(G)$, admitting an equivariantly closed extension $\eta_G \in \Omega^3_G(G)$ in the complex of equivariant differential forms. Our goal in this paper is to give an explicit, finite-dimensional description of an equivariant gerbe over G, with equivariant 3-curvature η_G.

A number of constructions of gerbes over compact Lie groups can be found in the literature, using different models of gerbes and valid in various degrees of generality. The differential geometry of gerbes was initiated by Brylinski's book [8], building on earlier work of Giraud. In this framework gerbes are viewed as sheafs of groupoids satisfying certain axioms. Brylinski gives a general construction of a gerbe with connection, for any integral closed 3-form on any 2-connected manifold M. The argument uses the path fibration $P_0M \to M$, and is similar to the well-known construction of a line bundle with connection out of a given integral closed 2-form on a simply connected manifold. In a later paper [9], Brylinski gives a finite-dimensional description of the sheaf of groupoids defining the basic gerbe for any compact Lie group G.

A less abstract picture, developed by Chatterjee-Hitchin [10, 18, 19], describes gerbes in terms of *transition line bundles* similar to the presentation of line bundles in terms of transition functions. A detailed construction of transition line bundles for the basic gerbe over $G = \mathrm{SU}(N)$ (as well as for the much more complicated case of finite quotients of $G = \mathrm{SU}(N)$) was obtained by Gawędzki-Reis [13].

In this paper, we will extend the Gawędzki-Reis approach from $\mathrm{SU}(N)$ to other simply connected simple Lie groups G. A fundamental difficulty in the more general case is that, in contrast to the case $G = \mathrm{SU}(N)$, the pull-back of a generator of $H^3_G(G, \mathbf{Z})$ to a conjugacy class $\mathcal{C} \subset G$ may not vanish. In this case it is impossible to describe the basic gerbe in terms of a G-invariant cover and G-equivariant transition line bundles. Compare with the case of G-equivariant line bundles over G-manifolds M: Such a line bundle may be described in terms of a G-invariant cover and G-invariant transition functions only if its pull-back to any G-orbit is equivariantly trivial.

One way of getting around this problem is to extend the Chatterjee-Hitchin theory to the equivariant case, as in [9, Appendix A]. A lift of the group action to a given gerbe is obtained by specifying the isomorphisms between the gerbe and its pull-back under the action of group elements $g \in G$. Unfortunately, the conditions for such isomorphisms to define a group action become rather complicated. A second possibility, adopted in this paper, is to use Murray's theory of *bundle gerbes* [24].

To explain our approach in more detail, let us first discuss the simplest case of $G = \mathrm{SU}(d+1)$, where it is equivalent to the construction in Gawędzki-Reis. The eigenvalues of any matrix $A \in \mathrm{SU}(d+1)$ can be uniquely written in the form

$$\exp(2\pi i \lambda_1(A)), \ldots, \exp(2\pi i \lambda_{d+1}(A))$$

where $\lambda_1(A), \ldots, \lambda_{d+1}(A) \in \mathbf{R}$ satisfy $\sum_{i=1}^{d+1} \lambda_i(A) = 0$ and

$$\lambda_1(A) \geq \lambda_2(A) \geq \ldots \geq \lambda_{d+1}(A) \geq \lambda_1(A) - 1\,.$$

Define an open cover $V_1, \ldots, V_d, V_{d+1}$ of G, where V_j consists of those matrices A for which the jth inequality becomes strict. Over the set G_{reg} of regular elements, where all inequalities are strict, we have $d+1$ line bundles $L_1, \ldots, L_d, L_{d+1}$ defined by the eigenlines for the eigenvalues $\exp(2\pi i \lambda_j(A))$. For $i < j$, the tensor product $L_{i+1} \otimes \cdots \otimes L_j \to G_{\mathrm{reg}}$ extends to a line bundle $L_{ij} \to V_i \cap V_j$. (One may view L_{ij} as the top exterior power of the sum of eigenspaces for the eigenvalues in the given range.) For $i < j < k$ we have a canonical isomorphism $L_{ij} \otimes L_{jk} \cong L_{ik}$ over the triple intersection $V_i \cap V_j \cap V_k$.

The L_{ij}, together with these isomorphisms, define a gerbe over $\mathrm{SU}(d+1)$, representing the generator of $H^3(\mathrm{SU}(d+1), \mathbf{Z})$.

More generally, consider any compact, simply connected, simple Lie group G of rank d. Up to conjugacy, G contains exactly $d+1$ elements with semi-simple centralizer. (For $G = \mathrm{SU}(d+1)$, these are the central elements.) Let $\mathcal{C}_1, \ldots, \mathcal{C}_{d+1} \subset G$ be their conjugacy classes. We will define an invariant open cover $V_1, \ldots, V_{d+1}$ of G, with the property that each member of this cover admits an equivariant retraction onto the conjugacy class $\mathcal{C}_j \subset V_j$. It turns out that every semi-simple centralizer has a distinguished central extension by $\mathrm{U}(1)$. This central extension defines an equivariant bundle gerbe on $\mathcal{C}_j$, hence (by pull-back) an equivariant bundle gerbe over V_j. We will find that these gerbes over V_j glue together to produce a gerbe over G, using a gluing rule developed in this paper.

The organization of the paper is as follows. In Section 2 we review the theory of gerbes and pseudo-line bundles with connections, and discuss 'strong equivariance' under a group action. Section 4 describes gluing rules for bundle gerbes. Section 3 summarizes some facts about gerbes coming from central extensions. In Section 5 we give the construction of the basic gerbe over G outlined above, and in Section 6 we study the 'pre-quantization of conjugacy classes'.

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2. Gerbes with connections

In this section we review gerbes on manifolds, along the lines of Chatterjee-Hitchin and Murray.

2.1 Chatterjee-Hitchin gerbes

Let M be a manifold. Any Hermitian line bundle over M can be described by an open cover U_a, and transition functions $\chi_{ab}\colon U_a \cap U_b \to \mathrm{U}(1)$ satisfying a cocycle condition $(\delta\chi)_{abc} = \chi_{bc}\chi_{ac}^{-1}\chi_{ab} = 1$ on triple intersections. The