by d^s_Γ, such a metric on Γ. We will call *mapping-telescope standard metric* any mapping-telescope d^s_Γ-metric on $\mathcal{C}(G_\alpha)$.

LEMMA 13.5. *The mapping-torus group G_α of an injective free group endomorphism acts cocompactly, properly discontinuously and isometrically on the Cayley complex $\mathcal{C}(G_\alpha)$ equipped with any mapping-telescope standard metric.*

Proof. We consider the usual action by left translations of the group on its Cayley graph. This action is extended in a natural way to a free action on the Cayley complex $\mathcal{C}(G_\alpha)$. Let f denote the map giving the strata for the structure of forest-stack of $\mathcal{C}(G_\alpha)$, see Lemma 13.3. For a mapping-telescope metric, all the strata $f^{-1}(r)$ and $f^{-1}(r+1)$ are isometric. And for a mapping-telescope standard metric all the strata $f^{-1}(n)$, $n \in \mathbf{Z}$, are equipped with the standard metric. This readily implies that the above action is isometric. □

13.2 FREE GROUP ENDOMORPHISMS AND FOREST-MAPS

The main point of Lemma 13.6 below is the so-called 'bounded-cancellation lemma' of [7] for free group automorphisms, and of [10] for the injective free group endomorphisms.

LEMMA 13.6. *Let α be an injective free group endomorphism. Let F and $\widetilde{\psi}$ be the forest and the forest-map on F given by Lemma 13.3. Then $\widetilde{\psi}$ is a weakly bi-Lipschitz forest-map of F equipped with the standard metric d^s_F.*

Proof. If w is any element in $F_n = \langle x_1, \dots, x_n \rangle$, and $|\,.\,|_{F_n}$ denotes the word-metric on F_n, then $|\alpha(w)|_{F_n} \leq (\max_{i=1,\dots,n} |\alpha(x_i)|_{F_n})|w|_{F_n}$. By definition of the standard metric, and setting $\mu_0 = \max_{i=1,\dots,n} |\alpha(x_i)|_{F_n}$, the map $\widetilde{\psi}$ satisfies $d^s_F(\widetilde{\psi}(x), \widetilde{\psi}(y)) \leq \mu_0 d^s_F(x, y)$ for any pair of *vertices* x, y. If x, y are not vertices, then they are joined in their stratum by a horizontal geodesic which is the concatenation of a path between two vertices, with two proper subsets of edges. By construction and simpliciality of $\widetilde{\psi}$, proper subsets of edges are dilated by a bounded factor when applying $\widetilde{\psi}$, so that the conclusion follows for the upper bound.

If w is any element in F_n then

$$\left|\alpha^{-1}(w)\right|_{F_n} \leq (\max_{i=1,\dots,n} \left|\alpha^{-1}(x_i)\right|_{F_n})|w|_{F_n}.$$

Setting $\mu_1 = \max_{i=1,\dots,n} \left|\alpha^{-1}(x_i)\right|_{F_n}$ we get $|\alpha(w)|_{F_n} \geq \frac{1}{\mu_1}|w|_{F_n}$. Therefore $d^s_F(\widetilde{\psi}(x), \widetilde{\psi}(y)) \geq \frac{1}{\mu_1} d^s_F(x, y)$ for any pair of *vertices* x, y. The inequality for all points x, y does not follow as easily as for the upper bound, since the map $\widetilde{\psi}$ might identify points, and this could make the distance decrease sharply. However, assume the existence of a constant K_0 such that $\widetilde{\psi}(x) = \widetilde{\psi}(y) \Rightarrow d^s_F(x,y) \leq K_0$. Any geodesic in F is the concatenation of a geodesic between two vertices with two proper subsets of edges of F. Thus the inequality $d^s_F(\widetilde{\psi}(x), \widetilde{\psi}(y)) \geq \frac{1}{\mu_1} d^s_F(x,y) - 2K_0$ follows in a straightforward way from the preceding assertions. Injective free group endomorphisms satisfy the so-called 'bounded-cancellation lemma' (see [10], and [7] for the particular case of automorphisms), i.e. there exists $A_\alpha > 0$ such that $|\alpha(w_1 w_2)|_{F_n} \geq |\alpha(w_1)|_{F_n} + |\alpha(w_2)|_{F_n} - A_\alpha$ for any w_1, w_2 in F_n with $|w_1 w_2|_{F_n} = |w_1|_{F_n} + |w_2|_{F_n}$. This inequality gives a constant $K_0 = A_\alpha + 2$ as required above, i.e. such that, if $\widetilde{\psi}(x) = \widetilde{\psi}(y)$ then $d^s_F(x,y) \leq K_0$. Setting $\mu = \max(\mu_0, \mu_1)$ and $K = 2K_0$, we get Lemma 13.6. □

LEMMA 13.7. *With the assumptions and notation of Lemma 13.6,*

1) *If α is hyperbolic then the forest-map is hyperbolic.*

2) *If α is hyperbolic and its image* $\mathrm{Im}(\alpha)$ *is malnormal, then the forest-map is strongly hyperbolic.*

Proof. (1) is easy to check. Let us prove (2). The notation used is that introduced in Section 13 when defining the forest F and the map $\widetilde{\psi}$. If the map is not strongly hyperbolic, there exists an infinite sequence of pairs of connected components (T_i, T'_i) such that T_i and T'_i are identified under $\widetilde{\psi}$ along a geodesic g_i and the length of g_i tends to $+\infty$ as $i \to +\infty$. Thus there exists an infinite number of elements $(u_i, u'_i) \in F_n - \mathrm{Im}(\alpha) \times F_n - \mathrm{Im}(\alpha)$ such that some geodesic word $a_i w_i b_i$ (resp. $a'_i w_i b'_i$) connects two vertices associated to elements in $u_i \mathrm{Im}(\alpha)$ (resp. in $u'_i \mathrm{Im}(\alpha)$) where the length of the w_i's tends to $+\infty$ as $i \to +\infty$.

Observe that in particular $a_i w_i b_i \in \mathrm{Im}(\alpha)$, $a'_i w_i b'_i \in \mathrm{Im}(\alpha)$, whereas $a_i w_i b'_i \notin \mathrm{Im}(\alpha)$ and $a'_i w_i b_i \notin \mathrm{Im}(\alpha)$ because they carry an element of $u_i \mathrm{Im}(\alpha)$ (resp. $u'_i \mathrm{Im}(\alpha)$) to an element of $u'_i \mathrm{Im}(\alpha)$ (resp. of $u_i \mathrm{Im}(\alpha)$). The lengths of the a_i, b_i, a'_i, b'_i can be assumed to be at most the maximum of the lengths of the images under α of the generators of F_n, which is finite. Since there are only a finite number of pairs of elements of bounded lengths, a same pair a_I, b_I (resp. a'_I, b'_I) appears an infinite number of times when listing the sequence of words $a_i w_i b_i$ (resp. $a'_i w_i b'_i$). The same finiteness argument then gives two words $\omega_1 \subsetneq \omega_2$ with $\omega_2 = \omega \omega_1$ such that $a_I \omega_j b_I \in \mathrm{Im}(\alpha)$, $a'_I \omega_j b'_I \in \mathrm{Im}(\alpha)$, $a_I \omega_j b'_I \notin \mathrm{Im}(\alpha)$ and $a'_I \omega_j b_I \notin \mathrm{Im}(\alpha)$, $j = 1, 2$.

Thus $a_I\omega_1 b_I b_I^{-1}\omega_1^{-1}\omega^{-1}a_I^{-1} \in \mathrm{Im}(\alpha)$, $a'_I\omega_1 b'_I b'^{-1}_I\omega_1^{-1}\omega^{-1}a'^{-1}_I \in \mathrm{Im}(\alpha)$, $a_I\omega_1 b'_I b'^{-1}_I\omega_1^{-1}\omega^{-1}a'^{-1}_I \notin \mathrm{Im}(\alpha)$. Now $(a_I\omega^{-1}a'^{-1}_I)^{-1} a_I\omega^{-1}a_I^{-1}(a_I\omega^{-1}a'^{-1}_I) = a'_I\omega^{-1}a'^{-1}_I \in \mathrm{Im}(\alpha)$, whereas $a_I\omega^{-1}a'^{-1}_I \notin \mathrm{Im}(\alpha)$ and $a_I\omega^{-1}a_I^{-1} \in \mathrm{Im}(\alpha)$. We thus get a contradiction to the malnormality of $\mathrm{Im}(\alpha)$ in F_n. This completes the proof. □

13.3 Proof of Theorem 13.2

From Lemmas 13.6 and 13.7, the Cayley complex $\mathcal{C}(G_\alpha)$ is the mapping-telescope of a strongly hyperbolic forest-map, equipped with the standard metric. A Cayley complex is connected. Thus, from Theorem 12.4, $\mathcal{C}(G_\alpha)$ is a Gromov-hyperbolic metric space for any mapping-telescope standard metric. From Lemma 13.5 the group G_α acts cocompactly, properly discontinuously and isometrically on $\mathcal{C}(G_\alpha)$ equipped with a mapping-telescope standard metric. A classical lemma of geometric group theory (usually attributed to Effremovich, Svàrc, Milnor – see [19] or [17] for instance), applied to quasi geodesic metric spaces, tells us that G_α and $\mathcal{C}(G_\alpha)$ are quasi-isometric so that G_α is a hyperbolic group. □

Remark 13.8. Another way of stating our main theorem about 'forest-stacks', using the language of trees of spaces, goes roughly as follows: "An oriented **R**-tree of **R**-trees with the gluing-maps satisfying the conditions of hyperbolicity and strong hyperbolicity with uniform constants is Gromov-hyperbolic." Here 'oriented **R**-tree' means an **R**-tree T equipped with an orientation going from the domain to the image of each attaching-map, and a surjective continuous map $f\colon T \to \mathbf{R}$ respecting this orientation. As a corollary of our theorem, and in order to illustrate it, we chose to concentrate on mapping-telescopes. We could as well consider spaces similar to mapping-telescopes but where we allow the attaching-maps not to be the same at each step. Our only requirement is to have uniform constants of quasi-isometry, hyperbolicity and so on. Also, with respect to groups, a corollary could have been stated dealing with HNN-extensions rather than just semi-direct products.

Another result which easily follows from our work could be more or less stated as follows. "Let T be a tree of spaces X_i, $i = 0, 1, \ldots$. Let $\psi\colon T \to T$ be a map of T such that the mapping-telescope of each X_i under ψ is Gromov-hyperbolic. If ψ induces a hyperbolic map on the tree resulting of the collapsing of each X_i to a point, then the mapping-telescope of the tree of spaces T under ψ is Gromov-hyperbolic." We leave the precise statement of such corollaries to the reader. Together with [14] where a new proof of the