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## 12.2 PROOF OF THEOREM 12.4

LEMMA 12.5. *Let  $(\Gamma, d_\Gamma)$  be an  $\mathbf{R}$ -forest. Let  $\psi$  be a weakly bi-Lipschitz forest-map of  $(\Gamma, d_\Gamma)$ . Let  $(K_\psi, f, \sigma_t)$  be the mapping-telescope of  $(\psi, \Gamma)$ , equipped with a structure of forest-stack as defined in Section 2. Then the semi-flow  $(\sigma_t)_{t \in \mathbf{R}^+}$  is a bounded-cancellation and bounded-dilatation semi-flow with respect to any horizontal  $d_\Gamma$ -metric (see Lemma 12.1).*

*Proof.* The horizontal metric  $\mathcal{H}$  agrees with the metric  $d_\Gamma$  on all the strata  $f^{-1}(n)$ ,  $n \in \mathbf{Z}$  (see Lemma 12.1). Consider any horizontal geodesic  $g$  in the stratum  $f^{-1}(0)$ . If  $\psi$  is weakly bi-Lipschitz with constants  $\mu_0$  and  $K_0$ , then for any integer  $n \geq 0$ , we have  $|[g]_n|_n \geq \frac{1}{\mu_0^n} |g|_0 - K_0 \left( \frac{1}{\mu_0^{n-1}} + \frac{1}{\mu_0^{n-2}} + \dots + 1 \right)$ . Since  $0 < \frac{1}{\mu_0} < 1$ , the sum tends to  $\frac{\mu_0}{\mu_0 - 1}$  as  $n \rightarrow +\infty$ . Setting  $\lambda_- = \frac{1}{\mu_0}$  and  $K = K_0 \frac{\mu_0}{\mu_0 - 1}$ , this proves the inequality of item (1) for horizontal geodesics in  $f^{-1}(n)$ ,  $n \in \mathbf{Z}$ , and an integer time  $t$ . For the case in which  $t$  is any positive real number and  $g \in f^{-1}(r)$ ,  $r$  any real number, just decompose  $\sigma_t = \sigma_{t-E[t]} \circ \sigma_{E[t-(E[r]+1-r)]} \circ \sigma_{E[r]+1-r}$ . The map  $\sigma_t$  is a homeomorphism from  $f^{-1}(r)$  onto  $f^{-1}(r+t)$  for any  $t \in [0, E[r]+1-r)$ . That is, for any real  $r$ ,  $|[g]_{r+t}|_{r+t} = |\sigma_t(g)|_{r+t}$  for  $t \in [0, E[r]+1-r)$ . The monotonicity of the maps  $l_{r,g}$  (see Lemma 12.1, item (2)) implies, for any  $r$  and  $t \in [0, E[r]+1-r)$ , that  $|\sigma_t(g)|_{r+t} \geq \frac{1}{\mu_0} |g|_r$ . The conclusion follows.  $\square$

LEMMA 12.6. *With the assumptions and notation of Lemma 12.5, if the map  $\psi$  is a (strongly) hyperbolic forest-map of  $(\Gamma, d_\Gamma)$  then the semi-flow  $(\sigma_t)_{t \in \mathbf{R}^+}$  is (strongly) hyperbolic with respect to any horizontal  $d_\Gamma$ -metric.*

The proof is similar to that of Lemma 12.5.  $\square$

*Proof of Theorem 12.4.* By Lemmas 12.5 and 12.6, a mapping-telescope admits a structure of forest-stack  $(\tilde{X}, f, \sigma_t, \mathcal{H})$  with horizontal metric  $\mathcal{H}$  such that the semi-flow  $(\sigma_t)_{t \in \mathbf{R}^+}$  is a strongly hyperbolic semi-flow with respect to  $\mathcal{H}$ . Hence Theorem 4.4 implies Theorem 12.4.  $\square$

## 13. ABOUT MAPPING-TORUS GROUPS

We first recall the definition of a *hyperbolic endomorphism* of a group introduced by Gromov [19].

DEFINITION 13.1 ([19], [3]). An injective endomorphism  $\alpha$  of the rank  $n$  free group  $F_n$  is hyperbolic if there exist  $\lambda_\alpha > 1$  and  $j_\alpha > 0$  such that for any  $w \in F_n$ , either  $\lambda_\alpha |w| \leq |\alpha^{j_\alpha}(w)|$  or  $w$  admits a preimage  $\alpha^{-j_\alpha}(w)$  such that  $\lambda_\alpha |w| \leq |\alpha^{-j_\alpha}(w)|$ , where  $|\cdot|$  denotes the usual word-metric.

We recall that a subgroup  $H$  in a group  $G$  is *malnormal* if  $w^{-1}Hw \cap H = \{1\}$  for any element  $w \notin H$  of  $G$ . We state our theorem about mapping-torus groups as follows:

THEOREM 13.2. *Let  $\alpha$  be an injective hyperbolic endomorphism of the rank  $n$  free group  $F_n$ . If the image of  $\alpha$  is a malnormal subgroup of  $F_n$  then the mapping-torus group  $G_\alpha = \langle x_1, \dots, x_n, t ; t^{-1}x_it = \alpha(x_i), i = 1, \dots, n \rangle$  is a hyperbolic group.*

### 13.1 RELATIONSHIPS WITH MAPPING-TELESCOPES

We consider the rank  $n$  free group  $F_n = \langle x_1, \dots, x_n \rangle$ . Let  $\alpha$  be an injective endomorphism of  $F_n$ . Let  $G_\alpha = \langle x_1, \dots, x_n, t ; t^{-1}x_it = \alpha(x_i), i = 1, \dots, n \rangle$  be the mapping-torus group of  $(\alpha, F_n)$ . We consider the Cayley graph  $\Gamma$  associated to the given system of generators. Let  $l$  be a loop in  $\Gamma$  whose associated word in the edges of  $\Gamma$  reads a relation  $t^{-1}x_it\alpha(x_i)^{-1}$ . We attach a 2-cell by its boundary circle along any such loop  $l$ . The resulting topological space is a 2-complex. This is the Cayley complex of the mapping-torus group  $G_\alpha$  for the given presentation.

Let us check that the above Cayley complex is a mapping-telescope of a forest-map. We consider the rose  $\mathcal{R}_n$  with  $n$  petals. We label each edge by a generator  $x_i$  of  $F_n$ . We denote by  $\psi$  the simplicial map on  $\mathcal{R}_n$  such that  $\psi(x_i)$  is a locally injective path whose associated word in the edges of  $\mathcal{R}_n$  reads  $\alpha(x_i)$ . Let us denote by  $T$  the universal covering of  $\mathcal{R}_n$  ( $T$  is a tree) and by  $\pi: T \rightarrow \mathcal{R}_n$  the associated covering-map. We denote by  $\hat{\psi}: T \rightarrow T$  a simplicial lift of  $\psi$  to  $T$ , that is  $\pi \circ \hat{\psi} = \psi \circ \pi$ . We consider the mapping-torus of  $(\psi, \mathcal{R}_n)$ , i.e. the 2-complex  $\mathcal{R}_n \times [0, 1]/(x, 1) \sim (\psi(x), 0)$ . Then the universal covering of this mapping-torus is the mapping-telescope of  $\tilde{\psi}: F \rightarrow F$ , where  $F$  and  $\tilde{\psi}$  are defined as follows:

- We denote by  $I$  the set of integers from 1 to  $\text{Card}(F_n/\text{Im}(\alpha))$ . The different classes are written  $w_i \text{Im}(\alpha)$ ,  $i = 0, 1, \dots$ . We denote by  $\gamma: I \rightarrow \{w_0, w_1, \dots\}$  the bijection. Then the connected components of  $F$  are in bijection with  $\mathbb{N}^{\text{Card}(I)}$ . Each connected component is the image, by a

bijection  $\mu$ , of a sequence of  $\text{Card}(I)$  integers. Each connected component  $\mu(x_0, x_1, \dots)$  of  $F$  is homeomorphic to  $T$  via  $\beta_{(x_0, x_1, \dots)}: \mu(x_0, x_1, \dots) \rightarrow T$ .

• We define the restriction of  $\tilde{\psi}$  to any connected component  $\mu((x_0, x_1, \dots))$  as follows:

If  $\text{Card}(I) < +\infty$  then

$$\tilde{\psi}|_{\mu((x_0, x_1, \dots))}: \begin{cases} \mu((x_0, x_1, \dots)) & \rightarrow \mu((E[\frac{x_0}{\text{Card}(I)}], x_1, \dots)) \\ x & \rightarrow (\gamma(j)\beta_{(x_0, x_1, \dots)}^{-1})\hat{\psi}\beta_{(x_0, x_1, \dots)}(x) \end{cases}$$

where  $j < \text{Card}(I)$  satisfies  $E[\frac{x_0}{\text{Card}(I)}] = k\text{Card}(I) + j$ .

If  $\text{Card}(I) = +\infty$  then

$$\tilde{\psi}|_{\mu((x_0, x_1, \dots))}: \begin{cases} \mu((x_0, x_1, \dots)) & \rightarrow \mu((x_1, x_2, \dots)) \\ x & \rightarrow (\gamma(x_0)\beta_{(x_0, x_1, \dots)}^{-1})\hat{\psi}\beta_{(x_0, x_1, \dots)}(x). \end{cases}$$

The mapping-torus of  $(\psi, \mathcal{R}_n)$  is a 2-complex whose 1-skeleton is the rose with  $n + 1$  petals in bijection with  $\{x_1, \dots, x_n, t\}$ . There is one 2-cell for each relation  $t^{-1}x_it\alpha(x_i)^{-1}$ . Thus the universal covering described above is the Cayley complex for  $G_\alpha$  with the presentation  $G_\alpha = \langle x_1, \dots, x_n, t; t^{-1}x_it = \alpha(x_i), i = 1, \dots, n \rangle$ . We have thus proved

LEMMA 13.3. *Let  $\alpha$  be an injective endomorphism of  $F_n = \langle x_1, \dots, x_n \rangle$ . Let  $G_\alpha = \langle x_1, \dots, x_n, t; t^{-1}x_it = \alpha(x_i), i = 1, \dots, n \rangle$  be the mapping-torus group of  $\alpha$ . Let  $\mathcal{C}(G_\alpha)$  be the Cayley complex of  $G_\alpha$  for the given presentation. Then  $\mathcal{C}(G_\alpha)$  is the mapping-telescope of a forest-map.*

REMARK 13.4. If the endomorphism  $\alpha$  is an automorphism then the above Cayley complex is the mapping-telescope of a tree-map. The tree is the universal covering of the rose with  $n$  petals. If the endomorphism  $\alpha$  is not injective then some element  $w \in F_n$  satisfies  $w = 1$  in  $G_\alpha$ ; the above construction fails because of the corresponding loops in the Cayley graph.

Let  $\alpha$  be an injective free group endomorphism. Let  $G_\alpha$  be the mapping-torus group of  $\alpha$ . Let  $\mathcal{C}(G_\alpha)$  be the Cayley complex of  $G_\alpha$  for the usual presentation  $G_\alpha = \langle x_1, \dots, x_n, t; t^{-1}x_it = \alpha(x_i), i = 1, \dots, n \rangle$ . By Lemma 13.3,  $\mathcal{C}(G_\alpha)$  is a mapping-telescope of a forest-map. We now want to see what happens with respect to metrics and dynamics. The Cayley graph of a group is equipped with a metric which makes each edge isometric to the interval  $(0, 1)$ . More generally, given a graph  $\Gamma$ , we call *standard metric*, and denote



by  $d_T^s$ , such a metric on  $\Gamma$ . We will call *mapping-telescope standard metric* any mapping-telescope  $d_T^s$ -metric on  $\mathcal{C}(G_\alpha)$ .

LEMMA 13.5. *The mapping-torus group  $G_\alpha$  of an injective free group endomorphism acts cocompactly, properly discontinuously and isometrically on the Cayley complex  $\mathcal{C}(G_\alpha)$  equipped with any mapping-telescope standard metric.*

*Proof.* We consider the usual action by left translations of the group on its Cayley graph. This action is extended in a natural way to a free action on the Cayley complex  $\mathcal{C}(G_\alpha)$ . Let  $f$  denote the map giving the strata for the structure of forest-stack of  $\mathcal{C}(G_\alpha)$ , see Lemma 13.3. For a mapping-telescope metric, all the strata  $f^{-1}(r)$  and  $f^{-1}(r+1)$  are isometric. And for a mapping-telescope standard metric all the strata  $f^{-1}(n)$ ,  $n \in \mathbb{Z}$ , are equipped with the standard metric. This readily implies that the above action is isometric.  $\square$

### 13.2 FREE GROUP ENDOMORPHISMS AND FOREST-MAPS

The main point of Lemma 13.6 below is the so-called ‘bounded-cancellation lemma’ of [7] for free group automorphisms, and of [10] for the injective free group endomorphisms.

LEMMA 13.6. *Let  $\alpha$  be an injective free group endomorphism. Let  $F$  and  $\tilde{\psi}$  be the forest and the forest-map on  $F$  given by Lemma 13.3. Then  $\tilde{\psi}$  is a weakly bi-Lipschitz forest-map of  $F$  equipped with the standard metric  $d_F^s$ .*

*Proof.* If  $w$  is any element in  $F_n = \langle x_1, \dots, x_n \rangle$ , and  $|\cdot|_{F_n}$  denotes the word-metric on  $F_n$ , then  $|\alpha(w)|_{F_n} \leq (\max_{i=1, \dots, n} |\alpha(x_i)|_{F_n}) |w|_{F_n}$ . By definition of the standard metric, and setting  $\mu_0 = \max_{i=1, \dots, n} |\alpha(x_i)|_{F_n}$ , the map  $\tilde{\psi}$  satisfies  $d_F^s(\tilde{\psi}(x), \tilde{\psi}(y)) \leq \mu_0 d_F^s(x, y)$  for any pair of vertices  $x, y$ . If  $x, y$  are not vertices, then they are joined in their stratum by a horizontal geodesic which is the concatenation of a path between two vertices, with two proper subsets of edges. By construction and simpliciality of  $\tilde{\psi}$ , proper subsets of edges are dilated by a bounded factor when applying  $\tilde{\psi}$ , so that the conclusion follows for the upper bound.

If  $w$  is any element in  $F_n$  then

$$|\alpha^{-1}(w)|_{F_n} \leq (\max_{i=1, \dots, n} |\alpha^{-1}(x_i)|_{F_n}) |w|_{F_n}.$$

Setting  $\mu_1 = \max_{i=1, \dots, n} |\alpha^{-1}(x_i)|_{F_n}$  we get  $|\alpha(w)|_{F_n} \geq \frac{1}{\mu_1} |w|_{F_n}$ . Therefore  $d_F^s(\tilde{\psi}(x), \tilde{\psi}(y)) \geq \frac{1}{\mu_1} d_F^s(x, y)$  for any pair of vertices  $x, y$ . The inequality

for all points  $x, y$  does not follow as easily as for the upper bound, since the map  $\tilde{\psi}$  might identify points, and this could make the distance decrease sharply. However, assume the existence of a constant  $K_0$  such that  $\tilde{\psi}(x) = \tilde{\psi}(y) \Rightarrow d_F^s(x, y) \leq K_0$ . Any geodesic in  $F$  is the concatenation of a geodesic between two vertices with two proper subsets of edges of  $F$ . Thus the inequality  $d_F^s(\tilde{\psi}(x), \tilde{\psi}(y)) \geq \frac{1}{\mu_1} d_F^s(x, y) - 2K_0$  follows in a straightforward way from the preceding assertions. Injective free group endomorphisms satisfy the so-called ‘bounded-cancellation lemma’ (see [10], and [7] for the particular case of automorphisms), i.e. there exists  $A_\alpha > 0$  such that  $|\alpha(w_1 w_2)|_{F_n} \geq |\alpha(w_1)|_{F_n} + |\alpha(w_2)|_{F_n} - A_\alpha$  for any  $w_1, w_2$  in  $F_n$  with  $|w_1 w_2|_{F_n} = |w_1|_{F_n} + |w_2|_{F_n}$ . This inequality gives a constant  $K_0 = A_\alpha + 2$  as required above, i.e. such that, if  $\tilde{\psi}(x) = \tilde{\psi}(y)$  then  $d_F^s(x, y) \leq K_0$ . Setting  $\mu = \max(\mu_0, \mu_1)$  and  $K = 2K_0$ , we get Lemma 13.6.  $\square$

LEMMA 13.7. *With the assumptions and notation of Lemma 13.6,*

- 1) *If  $\alpha$  is hyperbolic then the forest-map is hyperbolic.*
- 2) *If  $\alpha$  is hyperbolic and its image  $\text{Im}(\alpha)$  is malnormal, then the forest-map is strongly hyperbolic.*

*Proof.* (1) is easy to check. Let us prove (2). The notation used is that introduced in Section 13 when defining the forest  $F$  and the map  $\tilde{\psi}$ . If the map is not strongly hyperbolic, there exists an infinite sequence of pairs of connected components  $(T_i, T'_i)$  such that  $T_i$  and  $T'_i$  are identified under  $\tilde{\psi}$  along a geodesic  $g_i$  and the length of  $g_i$  tends to  $+\infty$  as  $i \rightarrow +\infty$ . Thus there exists an infinite number of elements  $(u_i, u'_i) \in F_n - \text{Im}(\alpha) \times F_n - \text{Im}(\alpha)$  such that some geodesic word  $a_i w_i b_i$  (resp.  $a'_i w_i b'_i$ ) connects two vertices associated to elements in  $u_i \text{Im}(\alpha)$  (resp. in  $u'_i \text{Im}(\alpha)$ ) where the length of the  $w_i$ 's tends to  $+\infty$  as  $i \rightarrow +\infty$ .

Observe that in particular  $a_i w_i b_i \in \text{Im}(\alpha)$ ,  $a'_i w_i b'_i \in \text{Im}(\alpha)$ , whereas  $a_i w_i b'_i \notin \text{Im}(\alpha)$  and  $a'_i w_i b_i \notin \text{Im}(\alpha)$  because they carry an element of  $u_i \text{Im}(\alpha)$  (resp.  $u'_i \text{Im}(\alpha)$ ) to an element of  $u'_i \text{Im}(\alpha)$  (resp. of  $u_i \text{Im}(\alpha)$ ). The lengths of the  $a_i, b_i, a'_i, b'_i$  can be assumed to be at most the maximum of the lengths of the images under  $\alpha$  of the generators of  $F_n$ , which is finite. Since there are only a finite number of pairs of elements of bounded lengths, a same pair  $a_I, b_I$  (resp.  $a'_I, b'_I$ ) appears an infinite number of times when listing the sequence of words  $a_i w_i b_i$  (resp.  $a'_i w_i b'_i$ ). The same finiteness argument then gives two words  $\omega_1 \subsetneq \omega_2$  with  $\omega_2 = \omega \omega_1$  such that  $a_I \omega_j b_I \in \text{Im}(\alpha)$ ,  $a'_I \omega_j b'_I \in \text{Im}(\alpha)$ ,  $a_I \omega_j b'_I \notin \text{Im}(\alpha)$  and  $a'_I \omega_j b_I \notin \text{Im}(\alpha)$ ,  $j = 1, 2$ .

Thus  $a_I \omega_1 b_I b_I^{-1} \omega_1^{-1} \omega^{-1} a_I^{-1} \in \text{Im}(\alpha)$ ,  $a'_I \omega_1 b'_I b_I'^{-1} \omega_1^{-1} \omega^{-1} a_I'^{-1} \in \text{Im}(\alpha)$ ,  $a_I \omega_1 b'_I b_I'^{-1} \omega_1^{-1} \omega^{-1} a_I'^{-1} \notin \text{Im}(\alpha)$ . Now  $(a_I \omega^{-1} a_I'^{-1})^{-1} a_I \omega^{-1} a_I^{-1} (a_I \omega^{-1} a_I'^{-1}) = a_I' \omega^{-1} a_I'^{-1} \in \text{Im}(\alpha)$ , whereas  $a_I \omega^{-1} a_I'^{-1} \notin \text{Im}(\alpha)$  and  $a_I \omega^{-1} a_I^{-1} \in \text{Im}(\alpha)$ . We thus get a contradiction to the malnormality of  $\text{Im}(\alpha)$  in  $F_n$ . This completes the proof.  $\square$

### 13.3 PROOF OF THEOREM 13.2

From Lemmas 13.6 and 13.7, the Cayley complex  $\mathcal{C}(G_\alpha)$  is the mapping-telescope of a strongly hyperbolic forest-map, equipped with the standard metric. A Cayley complex is connected. Thus, from Theorem 12.4,  $\mathcal{C}(G_\alpha)$  is a Gromov-hyperbolic metric space for any mapping-telescope standard metric. From Lemma 13.5 the group  $G_\alpha$  acts cocompactly, properly discontinuously and isometrically on  $\mathcal{C}(G_\alpha)$  equipped with a mapping-telescope standard metric. A classical lemma of geometric group theory (usually attributed to Effremovich, Svàrc, Milnor – see [19] or [17] for instance), applied to quasi geodesic metric spaces, tells us that  $G_\alpha$  and  $\mathcal{C}(G_\alpha)$  are quasi-isometric so that  $G_\alpha$  is a hyperbolic group.  $\square$

REMARK 13.8. Another way of stating our main theorem about ‘forest-stacks’, using the language of trees of spaces, goes roughly as follows: “An oriented  $\mathbf{R}$ -tree of  $\mathbf{R}$ -trees with the gluing-maps satisfying the conditions of hyperbolicity and strong hyperbolicity with uniform constants is Gromov-hyperbolic.” Here ‘oriented  $\mathbf{R}$ -tree’ means an  $\mathbf{R}$ -tree  $T$  equipped with an orientation going from the domain to the image of each attaching-map, and a surjective continuous map  $f: T \rightarrow \mathbf{R}$  respecting this orientation. As a corollary of our theorem, and in order to illustrate it, we chose to concentrate on mapping-telescopes. We could as well consider spaces similar to mapping-telescopes but where we allow the attaching-maps not to be the same at each step. Our only requirement is to have uniform constants of quasi-isometry, hyperbolicity and so on. Also, with respect to groups, a corollary could have been stated dealing with HNN-extensions rather than just semi-direct products.

Another result which easily follows from our work could be more or less stated as follows. “Let  $T$  be a tree of spaces  $X_i$ ,  $i = 0, 1, \dots$ . Let  $\psi: T \rightarrow T$  be a map of  $T$  such that the mapping-telescope of each  $X_i$  under  $\psi$  is Gromov-hyperbolic. If  $\psi$  induces a hyperbolic map on the tree resulting of the collapsing of each  $X_i$  to a point, then the mapping-telescope of the tree of spaces  $T$  under  $\psi$  is Gromov-hyperbolic.” We leave the precise statement of such corollaries to the reader. Together with [14] where a new proof of the

Bestvina-Feighn theorem is given for mapping-tori of surface groups, the last one gives, thanks to [26], a new proof of the full version of the Combination Theorem for mapping-tori of hyperbolic groups, namely: "If  $G$  is a hyperbolic group and  $\alpha$  is a hyperbolic automorphism of  $G$ , then  $G \rtimes_{\alpha} \mathbb{Z}$  is a hyperbolic group."

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