

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 49 (2003)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: HYPERBOLICITY OF MAPPING-TORUS GROUPS AND SPACES
Autor: Gautero, François
Kapitel: 12. Back to mapping-telescopes
DOI: <https://doi.org/10.5169/seals-66690>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 15.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Proposition 10.1 provides a $\kappa(r, s) = Bi(A(r, s), A(r, s))$ such that this bigon is $\kappa(r, s)$ -thin. Thus the given (r, s) -chain bigon is $\delta(r, s)$ -thin, with $\delta(r, s) = \kappa(r, s) + 2A(r, s)$. By Lemma 11.1, the given forest-stack, which is a $(1, 2)$ -quasi geodesic metric space, is $2\delta(1, 6)$ -hyperbolic. \square

12. BACK TO MAPPING-TELESCOPES

In this section we elucidate the relationships between forest-stacks and mapping-telescopes.

12.1 STATEMENT OF THE THEOREM

An **R-tree** (see [9], [2] among many others) is a metric space such that any two points are joined by a unique arc and this arc is a geodesic for the metric. In particular an **R-tree** is a topological tree. An **R-forest** is a union of disjoint **R-trees**.

LEMMA 12.1. *Let (Γ, d_Γ) be an **R-forest** and let $\psi: \Gamma \rightarrow \Gamma$ be a forest-map of Γ . Let (K_ψ, f, σ_t) be the mapping-telescope of (ψ, Γ) equipped with a structure of forest-stack as defined in Section 2. Then there is a horizontal metric $\mathcal{H} = (m_r)_{r \in \mathbf{R}}$ on K_ψ such that*

1. *The **R-forests** $(f^{-1}(r), m_r)$ and $(f^{-1}(r+1), m_{r+1})$ are isometric. Each stratum $(f^{-1}(n), m_n)$, $n \in \mathbf{Z}$, is isometric to (Γ, d_Γ) .*
2. *For any real r and any horizontal geodesic $g \in f^{-1}(r)$, the map*

$$l_{r,g}: \begin{cases} +1 - r] \rightarrow \mathbf{R}^+ \\ t \mapsto |\sigma_t(g)|_{r+t} \end{cases} .$$

is monotone.

Such a horizontal metric is called a horizontal d_Γ -metric. The telescopic metric associated to a horizontal d_Γ -metric is called a mapping-telescope d_Γ -metric.

Proof. We make each $\Gamma \times \{n\}$, $n \in \mathbf{Z}$, an **R-forest** isometric to Γ . We consider a cover of Γ by geodesics of length 1 which intersect only at their endpoints. Each $\Gamma \times \{n\}$ inherits the same cover. There is a disc $D_{e,n}$ in K_ψ for each such horizontal geodesic e in $\Gamma \times \{n\}$. This disc is bounded by e , $\psi(e)$ and the orbit-segments between the endpoints of e and those of $\psi(e)$.

We foliate this disc by segments with endpoints in, and transverse to, the orbit-segments in its boundary. Then we assign a length to each such segment so that the collection of lengths varies continuously and monotonically, from the length of e to that of $\psi(e)$. We thus obtain a horizontal metric on the mapping-telescope. Furthermore each stratum $f^{-1}(n)$, $n \in \mathbf{Z}$, is isometric to (Γ, d_Γ) . And the maps denoted by $l_{r,g}$ in Lemma 12.1 are monotone by construction. By definition of a mapping-telescope, the discs $D_{e,n}$ between $\Gamma \times \{n\}$ and $\Gamma \times \{n+1\}$ are copies of the discs $D_{e,n'}$ between $\Gamma \times \{n'\}$ and $\Gamma \times \{n'+1\}$, for any n, n' in \mathbf{Z} . This allows us to choose the horizontal metric to satisfy the further condition that $(f^{-1}(r), m_r)$ be isometric with $(f^{-1}(r+1), m_{r+1})$ for any real number r . \square

We now define dynamical properties for **R**-forest maps.

DEFINITION 12.2. Let (Γ, d_Γ) be an **R**-forest. A forest-map ψ of (Γ, d_Γ) is *weakly bi-Lipschitz* if there exist $\mu \geq 1$ and $K \geq 0$ such that $\mu d_\Gamma(x, y) \geq d_\Gamma(\psi(x), \psi(y)) \geq \frac{1}{\mu} d_\Gamma(x, y) - K$.

DEFINITION 12.3. Let (Γ, d_Γ) be an **R**-forest. A forest-map ψ of (Γ, d_Γ) is *hyperbolic* if it is weakly bi-Lipschitz and there exist $\lambda > 1$, $N \geq 1$, $M \geq 0$ such that for any pair of points x, y in Γ with $d_\Gamma(x, y) \geq M$, either $d_\Gamma(\psi^N(x), \psi^N(y)) \geq \lambda d_\Gamma(x, y)$ or $d_\Gamma(x_N, y_N) \geq \lambda d_\Gamma(x, y)$ for some x_N, y_N with $\psi^N(x_N) = x$, $\psi^N(y_N) = y$.

A hyperbolic forest-map ψ of (Γ, d_Γ) is *strongly hyperbolic* if, for any pair of points x, y with $d_\Gamma(x, y) \geq M$ and each connected component containing both a preimage of x and a preimage of y under ψ^N , there is at least one pair of such preimages x_N, y_N for which $d_\Gamma(x_N, y_N) \geq \lambda d_\Gamma(x, y)$.

If the forest Γ is a tree then a hyperbolic forest-map is strongly hyperbolic (similarly we saw that a hyperbolic semi-flow on a forest-stack whose strata are connected is strongly hyperbolic).

Our theorem about mapping-telescopes is

THEOREM 12.4. Let (Γ, d_Γ) be an **R**-forest. Let ψ be a strongly hyperbolic forest-map of (Γ, d_Γ) whose mapping-telescope K_ψ is connected. Then K_ψ is a Gromov-hyperbolic metric space for any mapping-telescope d_Γ -metric.

12.2 PROOF OF THEOREM 12.4

LEMMA 12.5. *Let (Γ, d_Γ) be an \mathbf{R} -forest. Let ψ be a weakly bi-Lipschitz forest-map of (Γ, d_Γ) . Let (K_ψ, f, σ_t) be the mapping-telescope of (ψ, Γ) , equipped with a structure of forest-stack as defined in Section 2. Then the semi-flow $(\sigma_t)_{t \in \mathbf{R}^+}$ is a bounded-cancellation and bounded-dilatation semi-flow with respect to any horizontal d_Γ -metric (see Lemma 12.1).*

Proof. The horizontal metric \mathcal{H} agrees with the metric d_Γ on all the strata $f^{-1}(n)$, $n \in \mathbf{Z}$ (see Lemma 12.1). Consider any horizontal geodesic g in the stratum $f^{-1}(0)$. If ψ is weakly bi-Lipschitz with constants μ_0 and K_0 , then for any integer $n \geq 0$, we have $|[g]_n|_n \geq \frac{1}{\mu_0^n} |g|_0 - K_0 \left(\frac{1}{\mu_0^{n-1}} + \frac{1}{\mu_0^{n-2}} + \dots + 1 \right)$. Since $0 < \frac{1}{\mu_0} < 1$, the sum tends to $\frac{\mu_0}{\mu_0 - 1}$ as $n \rightarrow +\infty$. Setting $\lambda_- = \frac{1}{\mu_0}$ and $K = K_0 \frac{\mu_0}{\mu_0 - 1}$, this proves the inequality of item (1) for horizontal geodesics in $f^{-1}(n)$, $n \in \mathbf{Z}$, and an integer time t . For the case in which t is any positive real number and $g \in f^{-1}(r)$, r any real number, just decompose $\sigma_t = \sigma_{t-E[r]} \circ \sigma_{E[r]-(E[r]+1-r)} \circ \sigma_{E[r]+1-r}$. The map σ_t is a homeomorphism from $f^{-1}(r)$ onto $f^{-1}(r+t)$ for any $t \in [0, E[r]+1-r)$. That is, for any real r , $|[g]_{r+t}|_{r+t} = |\sigma_t(g)|_{r+t}$ for $t \in [0, E[r]+1-r)$. The monotonicity of the maps $l_{r,g}$ (see Lemma 12.1, item (2)) implies, for any r and $t \in [0, E[r]+1-r)$, that $|\sigma_t(g)|_{r+t} \geq \frac{1}{\mu_0} |g|_r$. The conclusion follows. \square

LEMMA 12.6. *With the assumptions and notation of Lemma 12.5, if the map ψ is a (strongly) hyperbolic forest-map of (Γ, d_Γ) then the semi-flow $(\sigma_t)_{t \in \mathbf{R}^+}$ is (strongly) hyperbolic with respect to any horizontal d_Γ -metric.*

The proof is similar to that of Lemma 12.5. \square

Proof of Theorem 12.4. By Lemmas 12.5 and 12.6, a mapping-telescope admits a structure of forest-stack $(\tilde{X}, f, \sigma_t, \mathcal{H})$ with horizontal metric \mathcal{H} such that the semi-flow $(\sigma_t)_{t \in \mathbf{R}^+}$ is a strongly hyperbolic semi-flow with respect to \mathcal{H} . Hence Theorem 4.4 implies Theorem 12.4. \square

13. ABOUT MAPPING-TORUS GROUPS

We first recall the definition of a *hyperbolic endomorphism* of a group introduced by Gromov [19].