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Autor: Gautero, François
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Proposition 10.1 provides a $\kappa(r, s) = Bi(A(r, s), A(r, s))$ such that this bigon is $\kappa(r, s)$ -thin. Thus the given (r, s) -chain bigon is $\delta(r, s)$ -thin, with $\delta(r, s) = \kappa(r, s) + 2A(r, s)$. By Lemma 11.1, the given forest-stack, which is a $(1, 2)$ -quasi geodesic metric space, is $2\delta(1, 6)$ -hyperbolic. \square

12. BACK TO MAPPING-TELESCOPES

In this section we elucidate the relationships between forest-stacks and mapping-telescopes.

12.1 STATEMENT OF THE THEOREM

An **R-tree** (see [9], [2] among many others) is a metric space such that any two points are joined by a unique arc and this arc is a geodesic for the metric. In particular an **R-tree** is a topological tree. An **R-forest** is a union of disjoint **R-trees**.

LEMMA 12.1. *Let (Γ, d_Γ) be an **R-forest** and let $\psi: \Gamma \rightarrow \Gamma$ be a forest-map of Γ . Let (K_ψ, f, σ_t) be the mapping-telescope of (ψ, Γ) equipped with a structure of forest-stack as defined in Section 2. Then there is a horizontal metric $\mathcal{H} = (m_r)_{r \in \mathbf{R}}$ on K_ψ such that*

1. *The **R-forests** $(f^{-1}(r), m_r)$ and $(f^{-1}(r+1), m_{r+1})$ are isometric. Each stratum $(f^{-1}(n), m_n)$, $n \in \mathbf{Z}$, is isometric to (Γ, d_Γ) .*
2. *For any real r and any horizontal geodesic $g \in f^{-1}(r)$, the map*

$$l_{r,g}: \begin{cases}]-1-r] \rightarrow \mathbf{R}^+ \\ t \mapsto |\sigma_t(g)|_{r+t} \end{cases} .$$

is monotone.

Such a horizontal metric is called a horizontal d_Γ -metric. The telescopic metric associated to a horizontal d_Γ -metric is called a mapping-telescope d_Γ -metric.

Proof. We make each $\Gamma \times \{n\}$, $n \in \mathbf{Z}$, an **R-forest** isometric to Γ . We consider a cover of Γ by geodesics of length 1 which intersect only at their endpoints. Each $\Gamma \times \{n\}$ inherits the same cover. There is a disc $D_{e,n}$ in K_ψ for each such horizontal geodesic e in $\Gamma \times \{n\}$. This disc is bounded by e , $\psi(e)$ and the orbit-segments between the endpoints of e and those of $\psi(e)$.

We foliate this disc by segments with endpoints in, and transverse to, the orbit-segments in its boundary. Then we assign a length to each such segment so that the collection of lengths varies continuously and monotonically, from the length of e to that of $\psi(e)$. We thus obtain a horizontal metric on the mapping-telescope. Furthermore each stratum $f^{-1}(n)$, $n \in \mathbf{Z}$, is isometric to (Γ, d_Γ) . And the maps denoted by $l_{r,g}$ in Lemma 12.1 are monotone by construction. By definition of a mapping-telescope, the discs $D_{e,n}$ between $\Gamma \times \{n\}$ and $\Gamma \times \{n+1\}$ are copies of the discs $D_{e,n'}$ between $\Gamma \times \{n'\}$ and $\Gamma \times \{n'+1\}$, for any n, n' in \mathbf{Z} . This allows us to choose the horizontal metric to satisfy the further condition that $(f^{-1}(r), m_r)$ be isometric with $(f^{-1}(r+1), m_{r+1})$ for any real number r . \square

We now define dynamical properties for \mathbf{R} -forest maps.

DEFINITION 12.2. Let (Γ, d_Γ) be an \mathbf{R} -forest. A forest-map ψ of (Γ, d_Γ) is *weakly bi-Lipschitz* if there exist $\mu \geq 1$ and $K \geq 0$ such that $\mu d_\Gamma(x, y) \geq d_\Gamma(\psi(x), \psi(y)) \geq \frac{1}{\mu} d_\Gamma(x, y) - K$.

DEFINITION 12.3. Let (Γ, d_Γ) be an \mathbf{R} -forest. A forest-map ψ of (Γ, d_Γ) is *hyperbolic* if it is weakly bi-Lipschitz and there exist $\lambda > 1$, $N \geq 1$, $M \geq 0$ such that for any pair of points x, y in Γ with $d_\Gamma(x, y) \geq M$, either $d_\Gamma(\psi^N(x), \psi^N(y)) \geq \lambda d_\Gamma(x, y)$ or $d_\Gamma(x_N, y_N) \geq \lambda d_\Gamma(x, y)$ for some x_N, y_N with $\psi^N(x_N) = x$, $\psi^N(y_N) = y$.

A hyperbolic forest-map ψ of (Γ, d_Γ) is *strongly hyperbolic* if, for any pair of points x, y with $d_\Gamma(x, y) \geq M$ and each connected component containing both a preimage of x and a preimage of y under ψ^N , there is at least one pair of such preimages x_N, y_N for which $d_\Gamma(x_N, y_N) \geq \lambda d_\Gamma(x, y)$.

If the forest Γ is a tree then a hyperbolic forest-map is strongly hyperbolic (similarly we saw that a hyperbolic semi-flow on a forest-stack whose strata are connected is strongly hyperbolic).

Our theorem about mapping-telescopes is

THEOREM 12.4. *Let (Γ, d_Γ) be an \mathbf{R} -forest. Let ψ be a strongly hyperbolic forest-map of (Γ, d_Γ) whose mapping-telescope K_ψ is connected. Then K_ψ is a Gromov-hyperbolic metric space for any mapping-telescope d_Γ -metric.*

12.2 PROOF OF THEOREM 12.4

LEMMA 12.5. *Let (Γ, d_Γ) be an \mathbf{R} -forest. Let ψ be a weakly bi-Lipschitz forest-map of (Γ, d_Γ) . Let (K_ψ, f, σ_t) be the mapping-telescope of (ψ, Γ) , equipped with a structure of forest-stack as defined in Section 2. Then the semi-flow $(\sigma_t)_{t \in \mathbf{R}^+}$ is a bounded-cancellation and bounded-dilatation semi-flow with respect to any horizontal d_Γ -metric (see Lemma 12.1).*

Proof. The horizontal metric \mathcal{H} agrees with the metric d_Γ on all the strata $f^{-1}(n)$, $n \in \mathbf{Z}$ (see Lemma 12.1). Consider any horizontal geodesic g in the stratum $f^{-1}(0)$. If ψ is weakly bi-Lipschitz with constants μ_0 and K_0 , then for any integer $n \geq 0$, we have $|[g]_n|_n \geq \frac{1}{\mu_0^n} |g|_0 - K_0 \left(\frac{1}{\mu_0^{n-1}} + \frac{1}{\mu_0^{n-2}} + \dots + 1 \right)$. Since $0 < \frac{1}{\mu_0} < 1$, the sum tends to $\frac{\mu_0}{\mu_0 - 1}$ as $n \rightarrow +\infty$. Setting $\lambda_- = \frac{1}{\mu_0}$ and $K = K_0 \frac{\mu_0}{\mu_0 - 1}$, this proves the inequality of item (1) for horizontal geodesics in $f^{-1}(n)$, $n \in \mathbf{Z}$, and an integer time t . For the case in which t is any positive real number and $g \in f^{-1}(r)$, r any real number, just decompose $\sigma_t = \sigma_{t-E[t]} \circ \sigma_{E[t-(E[r]+1-r)]} \circ \sigma_{E[r]+1-r}$. The map σ_t is a homeomorphism from $f^{-1}(r)$ onto $f^{-1}(r+t)$ for any $t \in [0, E[r]+1-r)$. That is, for any real r , $|[g]_{r+t}|_{r+t} = |\sigma_t(g)|_{r+t}$ for $t \in [0, E[r]+1-r)$. The monotonicity of the maps $l_{r,g}$ (see Lemma 12.1, item (2)) implies, for any r and $t \in [0, E[r]+1-r)$, that $|\sigma_t(g)|_{r+t} \geq \frac{1}{\mu_0} |g|_r$. The conclusion follows. \square

LEMMA 12.6. *With the assumptions and notation of Lemma 12.5, if the map ψ is a (strongly) hyperbolic forest-map of (Γ, d_Γ) then the semi-flow $(\sigma_t)_{t \in \mathbf{R}^+}$ is (strongly) hyperbolic with respect to any horizontal d_Γ -metric.*

The proof is similar to that of Lemma 12.5. \square

Proof of Theorem 12.4. By Lemmas 12.5 and 12.6, a mapping-telescope admits a structure of forest-stack $(\tilde{X}, f, \sigma_t, \mathcal{H})$ with horizontal metric \mathcal{H} such that the semi-flow $(\sigma_t)_{t \in \mathbf{R}^+}$ is a strongly hyperbolic semi-flow with respect to \mathcal{H} . Hence Theorem 4.4 implies Theorem 12.4. \square

13. ABOUT MAPPING-TORUS GROUPS

We first recall the definition of a *hyperbolic endomorphism* of a group introduced by Gromov [19].