**Zeitschrift:** L'Enseignement Mathématique
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique
**Band:** 49 (2003)
**Heft:** 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** HYPERBOLICITY OF MAPPING-TORUS GROUPS AND SPACES
**Autor:** Gautero, François
**Kapitel:** 9. PUTTING PATHS IN FINE POSITION
**DOI:** https://doi.org/10.5169/seals-66690

**Nutzungsbedingungen**
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

**Conditions d'utilisation**
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

**Terms of use**
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

**Download PDF:** 15.02.2026

**ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch**

geodesic, with the same endpoints, in fine position with respect to $h$, which is $C_{7.3}(A(|h|_r, J, J'), J, J')$-close to $g$ and which is a stair or the concatenation of two stairs. Lemma 6.4, together with Lemma 5.4 applied as above, then provide $C_{6.4}(B(|h|_r, J, J'), B(|h|_r, J, J'))$ and

$$D(|h|_r, J, J') = C_{5.4}(1, 3, C_{6.4}(B(|h|_r, J, J'), B(|h|_r, J, J'))$$

such that this, or these, stair(s) are $D(|h|_r, J, J')$-close to the orbit-segments between $h$ and their endpoints. We conclude that $g$ is $C_{7.3}(A(|h|_r, J, J'), J, J') + D(|h|_r, J, J')$-close to these orbit-segments. The last point of the proposition is obvious. □

## 9. Putting paths in fine position

Proposition 9.1. *Let $h$ be a horizontal geodesic. Let $g$ be a straight $(J, J')$-quasi geodesic, which joins the future or past orbits of the endpoints of $h$. There exist a constant $C_{9.1}(J, J')$ and a $(C_{9.1}(J, J'), C_{9.1}(J, J'))$-quasi geodesic $\mathcal{G}$ which is $C_{9.1}(J, J')$-close to $g$, which has the same endpoints as $g$, and which is in fine position with respect to $h$.*

*Proof.* We consider a maximal subpath $g'$ of $g$ whose endpoints lie in the future or past orbits of some points in $h$, and such that no other point of $g'$ satisfies this property. Consider any maximal $-$-hole $b$ in $g'$, and let $I$ denote the horizontal geodesic between the endpoints of $b$.

Case 1. Either $I$ is contained in a cancellation or $I$ is the concatenation of two horizontal geodesics, each contained in a cancellation.

Lemma 6.7 gives $C_{6.7}(J, J')$ such that, if $|I|_{f(I)} \geq C_{6.7}(J, J')$ then $I$ is dilated in the future after $C_{6.7}(J, J')t_0$. Lemma 5.3 gives $C_{5.3}(C_{6.7}(J, J'))$ such that the horizontal length of any horizontal geodesic contained in a cancellation and dilated in the future after $C_{6.7}(J, J')t_0$ is at most $C_{5.3}(C_{6.7}(J, J'))$. By Lemma 5.4 we get an upper bound $C_{5.4}(C_{6.7}(J, J'), 2, C_{5.3}(C_{6.7}(J, J')))$ on the horizontal length of $I$.

Case 2. There exists another horizontal geodesic in another connected component of the same stratum whose pulled-tight projection agrees with that of $I$ after some finite time.

We consider the maximal geodesic preimage $I'$ of $I$ under $\sigma_{C_{6.7}(J,J')t_0}$ which connects two points of $b$. It admits a decomposition into subpaths $I'_\alpha$

connecting points in $b$ such that the subpath of $b$ between the endpoints of each $I'_\alpha$ is a $-$-hole. The strong hyperbolicity of the semi-flow implies, by Lemma 6.7, that the horizontal length of each $I'_\alpha$ is bounded above by $C_{6.7}(J,J')$. Since $g$ is a $(J,J')$-quasi geodesic, we get $\max_{x\in b}(f(I)-f(x)) \leq JC_{6.7}(J,J')+J'+C_{6.7}(J,J')$.

CASE 3. Some subpath of $I$ connects the future or past orbits of points in $h$.

The only possibility is that $I$ be a pulled-tight image of $h$, i.e. $g' = b$. Consider a geodesic preimage $I'$ of $I$ under $\sigma_{C_{6.7}(J,J')t_0}$ between two points in $b$. Then proceed as in Case 2, the only difference being that for each subpath $I_\alpha$, *either* there exists a horizontal geodesic in another connected component of the same stratum, whose pulled-tight projection agrees with that of $I_\alpha$ after some finite time (this is exactly Case 2), *or* $I_\alpha$ is contained in a cancellation or in the union of two cancellations, and the arguments are exactly those of Case 1. The bounded-dilatation property then gives an upper bound on the horizontal length of $I$.

We denote by $A(J,J')$ the largest of the constants found in Cases 1, 2 and 3. We denote by $A'(J,J')$ the largest of the constants $A(J,J')$, $C_{7.3}(A(J,J'),J,J')$ and $C_{7.2}(A(J,J'),J,J')$. Lemmas 7.2, 7.3 and 7.1 then give $B(J,J') = C_{7.1}(A'(J,J'),A'(J,J'),\ J,J')$, such that replacing the maximal $-$-holes in $g'$ by the horizontal geodesic between their endpoints yields a straight $(B(J,J'),B(J,J'))$-quasi geodesic stair $S$, with the same endpoints, which is $A'(J,J')$-close to $g'$. Let $I'$ be a horizontal geodesic between $S$ and a future or past orbit of some point in $h$, which is minimal in the sense of inclusion, i.e. does not contain any subpath connecting $S$ to a future or past orbit of a point in $h$. This horizontal geodesic $I'$ is a pulled-tight image of a subpath of $S$ in the stratum considered. It is either contained in a cancellation, or is the union of two horizontal geodesics contained in a cancellation. Lemma 6.4 gives $C_{6.4}(B(J,J'),B(J,J'))$ such that, if $|I'|_{f(I')} \geq C_{6.4}(B(J,J'),B(J,J'))$ then $I'$ is dilated in the futur after $t_0$. From Lemmas 5.3 and 5.4 we get $|I'|_{f(I')} \leq C_{5.4}(1,2,C_{5.3}(1))$. Therefore $S$ is at horizontal distance at most $D(J,J') = \max(C_{6.4}(B(J,J'),B(J,J')),C_{5.4}(1,2,C_{5.3}(1)))$ from a straight stair $\mathcal{S}(g')$, with the same endpoints and in fine position with respect to $h$. Lemmas 7.4 and 7.1 then give $E(J,J') = C_{7.1}(C_{7.4}(D(J,J'),B(J,J'),B(J,J')),C_{7.4}(D(J,J'),B(J,J'),B(J,J')),J,J')$ such that replacing the maximal subpaths $g'$ as above by the given stair $\mathcal{S}(g')$ gives a straight $(E(J,J'),E(J,J'))$-quasi geodesic, with the same endpoints as $g$, in fine position with respect to $h$, and which is $D(J,J')$-close to $g$. $\square$