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Hence $|I_C|_{f(I)} \geq \frac{\lambda^n - \lambda_+^{-n}}{\lambda^n - \lambda_-^{-n}} |I_D|_{f(I)}$, so that $|I_C|_{f(I)} \geq \frac{X(n)}{1+X(n)} |I|_{f(I)}$ with $X(n) = \frac{\lambda^n - \lambda_+^{-n}}{\lambda^n - \lambda_-^{-n}}$. Now $\lim_{n \rightarrow +\infty} \frac{X(n)}{1+X(n)} = 1$, so that for some $n_* \geq 1$, for any $n \geq n_*$, $\frac{X(n)}{1+X(n)} \geq \frac{1}{2}$. Since the horizontal length of any interval I_k in I_C is at most $C_{6.2}(J, J')$, and the telescopic length of the associated $p_k \subset p$ is at least t_0 , we obtain

$$|p|_{(\tilde{X}, \mathcal{H})} \geq \frac{t_0}{2C_{6.2}(J, J')} |I|_{f(I)}.$$

On the other hand, $|p|_{(\tilde{X}, \mathcal{H})} \leq 2Jnt_0 + \lambda^{-n}J |I|_{f(I)} + J'$ for any $n \geq n_*$. The last two inequalities give, for $n \geq n_*$, $2Jnt_0 + \lambda^{-n}J |I|_{f(I)} + J' \geq \frac{t_0}{2C_{6.2}(J, J')} |I|_{f(I)}$, equivalently $2Jnt_0 + J' \geq (\frac{t_0}{2C_{6.2}(J, J')} - \lambda^{-n}J) |I|_{f(I)}$. We choose $n_o \geq n_*$ such that $\frac{t_0}{2C_{6.2}(J, J')} - \lambda^{-n_o}J > 0$. We get

$$\frac{2Jn_o t_0 + J'}{\frac{t_0}{2C_{6.2}(J, J')} - \lambda^{-n_o}J} \geq |I|_{f(I)}.$$

Thus, for $|I|_{f(I)} > \frac{2Jn_o t_0 + J'}{\frac{t_0}{2C_{6.2}(J, J')} - \lambda^{-n_o}J}$, h is not dilated in the future after t_0 . If $|I|_{f(I)} > \lambda_+^{n_o}M$, then $|h|_{f(h)} \geq M$. Therefore h is dilated in the past after t_0 . We choose N such that $\lambda^N \lambda_+^{-n_o} > \lambda$. Thus, if $|I|_{f(I)} \geq \max(\lambda_+^{n_o}M, \frac{2Jn_o t_0 + J'}{\frac{t_0}{2C_{6.2}(J, J')} - \lambda^{-n_o}J})$ then I is dilated in the past after $(n_o C_{6.2}(J, J') + N)t_0$. The arguments and computations in the case where $\max_{x \in p} f(x) \leq f(I)$ are the same. \square

7. SUBSTITUTION OF QUASI GEODESICS

LEMMA 7.1. *Let p be a (J, J') -quasi geodesic. Let q be obtained from p by replacing subpaths $p_i \subset p$ by (L, L') -quasi geodesics q_i satisfying the following properties:*

- q_i has the same endpoints as p_i ,
- q_i is L -close to p_i ,
- $|q_i|_{(\tilde{X}, \mathcal{H})} \leq L|p_i|_{(\tilde{X}, \mathcal{H})}$.

There exists a constant $C_{7.1}(L, L', J, J')$, which increases in each variable, such that q is a $(C_{7.1}(L, L', J, J'), C_{7.1}(L, L', J, J'))$ -quasi geodesic which is L -close to p .

Proof. Since each q_i is L -close to a p_i , and with the same endpoints, q is L -close to p . Let us consider any two points x, y in q and let $q_{xy} \subset q$

be the subpath of q between x and y . If both x and y lie in a q_i , or in a same subpath in the closed complement of the union of the q_i 's, then $|q_{xy}|_{(\tilde{X}, \mathcal{H})} \leq \max(L, J)d_{(\tilde{X}, \mathcal{H})}(x, y) + \max(L', J')$. Otherwise $q_{xy} = w_1 w_2 w_3$, where w_1, w_3 are contained either in some q_i or in p , and w_2 begins and ends with the initial or terminal point of some q_i . The third property concerning the q_i 's leads to $|w_2|_{(\tilde{X}, \mathcal{H})} \leq L|p_2|_{(\tilde{X}, \mathcal{H})}$, where $p_2 \subset p$ is the subpath of p with the same endpoints as w_2 . Thus $|q_{xy}|_{(\tilde{X}, \mathcal{H})} \leq LJd_{(\tilde{X}, \mathcal{H})}(x, y) + 2 \max(L', LJ')$. \square

LEMMA 7.2. *Let p be a straight (J, J') -quasi geodesic $-$ hole such that $\max_{x \in p} f(I) - f(x) \leq L$, where I is the horizontal geodesic joining the endpoints of p . Then there exists a constant $C_{7.2}(L, J, J') \geq M$, which increases in each variable, such that*

$$1) \quad |I|_{f(I)} \leq C_{7.2}(L, J, J')|p|_{(\tilde{X}, \mathcal{H})}.$$

2) I is a straight $(C_{7.2}(L, J, J'), C_{7.2}(L, J, J'))$ -quasi geodesic which is $C_{7.2}(L, J, J')$ -close to p .

Proof. A horizontal geodesic is always straight. The horizontal geodesic I is the pulled-tight projection of p . Thus, by the bounded-dilatation property, $|I|_{f(I)} \leq \lambda_+^L |p|_{(\tilde{X}, \mathcal{H})}$. By Lemma 5.6, I is $C_{5.6}(L)$ -close to p . Consider any subpath I' of I ; it is the pulled-tight projection of some subpath p' of p . By the bounded-dilatation property, $|I'|_{f(I)} \leq \lambda_+^L |p'|_{(\tilde{X}, \mathcal{H})}$. Since p is a (J, J') -quasi geodesic, $|I'|_{f(I)} \leq \lambda_+^L (Jd_{(\tilde{X}, \mathcal{H})}(i(p'), t(p')) + J')$. Since I' is $C_{5.6}(L)$ -close to p' , $|I'|_{f(I)} \leq \lambda_+^L Jd_{(\tilde{X}, \mathcal{H})}(i(I'), t(I')) + \lambda_+^L (2JC_{5.6}(L) + J')$. \square

LEMMA 7.3. *Let p be a straight (J, J') -quasi geodesic $-$ hole such that the horizontal length of the horizontal geodesic I between its endpoints is less than or equal to L . Then there exists a constant $C_{7.3}(L, J, J') \geq M$, which increases in each variable, such that*

$$1) \quad |I|_{f(I)} \leq C_{7.3}(L, J, J')|p|_{(\tilde{X}, \mathcal{H})}.$$

2) I is a straight $(C_{7.3}(L, J, J'), C_{7.3}(L, J, J'))$ -quasi geodesic which is $C_{7.3}(L, J, J')$ -close to p .

Proof. Since p is a (J, J') -quasi geodesic,

$$\max_{x \in p} |f(x) - f(I)| \leq J|I|_{f(I)} + J'.$$

Lemma 7.3 now follows from Lemma 7.2. \square

LEMMA 7.4. *Let p be a straight (J, J') -quasi geodesic stair. For any $L \geq 0$, there exists a constant $C_{7.4}(L, J, J')$, which increases in each variable, such that if q is a straight stair whose points are at horizontal distance at most L from p , and with the same endpoints as p , then*

1) q is a straight $(C_{7.4}(L, J, J'), C_{7.4}(L, J, J'))$ -quasi geodesic stair which is L -close to p .

2) $|q|_{(\tilde{X}, \mathcal{H})} \leq C_{7.4}(L, J, J')|p|_{(\tilde{X}, \mathcal{H})}$.

Proof. Consider a stair S , in the disc bounded by $p \cup q$, whose endpoints are those of p and q , and whose vertical geodesics end at q , all the stairs being oriented so that f is increasing along them. Consider a subpath S' of S which is the concatenation of a vertical segment followed by a horizontal one. By assumption, the horizontal length X of S' is bounded above by L . Let t be its vertical length. The bounded-dilatation property implies that the quotient of $|S'|_{(\tilde{X}, \mathcal{H})}$ by the telescopic length of the subpath of p between the endpoints of S' is bounded above by $Q = \frac{t+X}{t+\lambda_+^{-1}X}$. Since $X \leq L$, Q tends to 1 as $t \rightarrow +\infty$. One thus obtains a constant T such that for $t \geq T$, Q is bounded above by some constant, depending on L . When both t and X are close to 0 then Q is close to 1. Hence, since Q is continuous, Q admits an upper bound, denoted by $A(L)$, for all the t and X considered. This upper bound will be the same for all the subpaths S' as above.

The stair S is a concatenation of such subpaths S' , possibly with one or two subpaths of p at the extremities. Thus the additivity of the telescopic length gives $|S|_{(\tilde{X}, \mathcal{H})} \leq A(L)|p|_{(\tilde{X}, \mathcal{H})}$. Let S'' be a subpath of S which is the concatenation of a horizontal subpath followed by a vertical one. The path S is the concatenation of such subpaths S'' with possibly one or two subpaths of q at the extremities. Exactly the same arguments as above give $|q|_{(\tilde{X}, \mathcal{H})} \leq A(L)|S|_{(\tilde{X}, \mathcal{H})}$. We thus get $|q|_{(\tilde{X}, \mathcal{H})} \leq A(L)^2|p|_{(\tilde{X}, \mathcal{H})}$. It only remains to prove that q is a quasi geodesic with constants of quasi geodesicity depending only on L, J, J' . Let x, y be any two points in q . As usual q_{xy} is the subpath of q between x and y and we denote by $p_{x'y'}$ the subpath of p between the two points x', y' in p which are at horizontal distance at most L from x and y . We consider a stair S between q_{xy} and $p_{x'y'}$, with the same endpoints as q_{xy} . The same arguments as above apply and give $|q_{xy}|_{(\tilde{X}, \mathcal{H})} \leq A(L)^2|p_{x'y'}|_{(\tilde{X}, \mathcal{H})}$. Since p is a (J, J') -quasi geodesic, we conclude that $|q_{xy}|_{(\tilde{X}, \mathcal{H})} \leq JA(L)^2d_{(\tilde{X}, \mathcal{H})}(x', y') + J'A(L)^2$. Since $d_{(\tilde{X}, \mathcal{H})}(x', y') \leq d_{(\tilde{X}, \mathcal{H})}(x, y) + 2L$, the proof of Lemma 7.4 is complete. \square