L'Enseignement Mathématique, t. 49 (2003), p. 263–305

HYPERBOLICITY OF MAPPING-TORUS GROUPS AND SPACES

by François GAUTERO

ABSTRACT. This paper deals with the geometry of metric 'two-dimensional' spaces, equipped with semi-flows admitting transverse foliations by forests. Our main theorem relates the Gromov-hyperbolicity of such spaces, for instance mapping-telescopes of $\mathbf{R}$-trees, with the dynamical behaviour of the semi-flow. As a corollary, we give a new proof of the following theorem [3]: *Let α be a hyperbolic injective endomorphism of the rank n free group F_n. If the image of α is a malnormal subgroup of F_n, then $G_\alpha = F_n \rtimes_\alpha \mathbf{Z}$ is a hyperbolic group.*

INTRODUCTION

The subject of 3-dimensional topology changed completely in the seventies with Thurston's geometric methods. His geometrization conjecture involves eight classes of manifolds, among which the hyperbolic manifolds play the most important role. In this context, a hyperbolic manifold is a compact manifold which admits (or whose interior admits in the case of non-empty boundary) a metric of constant curvature -1. According to another conjecture of Thurston, any closed hyperbolic 3-manifold should have a finite cover which is a mapping-torus. This gives a particular interest to these mapping-tori manifolds. Recall that a mapping-torus is a manifold which fibers over the circle. Namely this is a 3-manifold constructed from a homeomorphism h of a compact surface Σ as

$$M = (\Sigma \times [0,1])/((x,1) \sim (h(x),0)).$$

For these manifolds, the hyperbolization conjecture has been proved, see for instance [25]: the manifold M constructed from Σ and h as above is hyperbolic if and only if Σ has negative Euler characteristic and h is a pseudo-Anosov homeomorphism (see [12]).

In parallel to these developments in 3-dimensional topology, there has been a revival in combinatorial group theory. First introduced by Dehn at the beginning of the twentieth century, geometric methods were reintroduced in this field by Gromov in the 80's. The notion of hyperbolicity carries over in some sense from manifolds to metric spaces and groups. We then speak of Gromov hyperbolicity. Such metric spaces and groups are also called weakly hyperbolic, or negatively curved, or word-hyperbolic, see [19] as well as [16], [1], [8] or [5] among others. Mapping-tori manifolds have the following analogue in this setting: given a finitely presented group $F = \langle S ; R \rangle$, $S = \{x_1, \ldots, x_n\}$, and an endomorphism α of F, the mapping-torus group of (α, F) is the group with presentation $\langle x_1, \ldots, x_n, t ; R, t^{-1}x_i t = \alpha(x_i), i = 1, \ldots, n \rangle$. For instance, if the 3-manifold M is the mapping-torus of (h, Σ) and if $h_\#$ is the automorphism induced by h on the fundamental group of Σ, then the fundamental group of M is the mapping-torus group of $(h_\#, \pi_1(\Sigma))$. In fact, in this case, since $h_\#$ is an automorphism of $\pi_1(\Sigma)$, the mapping-torus group is easily described as the semi-direct product $\pi_1(\Sigma) \rtimes_{h_\#} \mathbf{Z}$.

The main and central result in group theory concerning the preservation of hyperbolicity under extension is the Combination Theorem of [3] (see also a clear exposition of this theorem in [20]). Alternative proofs have been presented since the original paper of Bestvina-Feighn ([18], [22]), but concerning essentially the so-called 'acylindrical case', where the 'Annuli Flare Condition' of [3] is vacuously satisfied. Gersten [15] proves a converse of the Combination Theorem. At the periphery of this theorem, let us also cite [11] and [24] about the hyperbolicity of other kinds of extensions or [23], which shows the existence of Cannon-Thurston maps in this context.

As a corollary of the Combination Theorem, and to illustrate it, the authors of [3] emphasize the following result: Let F be a hyperbolic group and let α be an automorphism of F. Assume that α is hyperbolic, namely that there exist $m \in \mathbf{Z}$ and $\lambda \in \mathbf{R}$, $\lambda > 1$, such that for any element f of word-length $l(f)$ in the generators of F, we have $\max\big(l(\alpha^m(f)), l(\alpha^{-m}(f))\big) \geq \lambda l(f)$. Then $F \rtimes_\alpha \mathbf{Z}$ is a hyperbolic group. This corollary lives in a different world than the above mentioned alternative proofs of the Combination Theorem, namely it is 'non-acylindrical'. No paper, except the original one of Bestvina-Feighn, covers it. Swarup used it to give a weak hyperbolization theorem for 3-manifolds [27]. Hyperbolic automorphisms were defined by Gromov [19], see also [3]. From [26], if a hyperbolic automorphism is defined on a hyperbolic group then this hyperbolic group is the free product of two kinds of groups: free groups and fundamental groups of closed surfaces with negative Euler characteristic. Hyperbolic automorphisms of fundamental groups of closed surfaces are exactly the automorphisms induced by pseudo-Anosov homeomorphisms. Brinkmann characterized the hyperbolic automorphisms of free groups as the automorphisms without any finite invariant set of conjugacy-classes [6]. Below we consider hyperbolic injective free group endomorphisms. The notion of hyperbolic automorphism is generalized in a straightforward way to injective endomorphisms. We give a new proof of the Bestvina-Feighn theorem in this setting:

THEOREM 0.1. *Let $F_n = \langle x_1, \dots, x_n \rangle$ be the free group of rank n. Let α be a hyperbolic injective endomorphism of F_n. Assume that the image of α is malnormal, that is $w^{-1}\,\mathrm{Im}(\alpha) w \cap \mathrm{Im}(\alpha) = \{1\}$ for any $w \notin \mathrm{Im}(\alpha)$ of F_n. Then the mapping-torus group $G_\alpha = \big\langle x_1, \dots, x_n, t\,;\, t^{-1}x_it = \alpha(x_i),\ i = 1, \dots, n \big\rangle$ is a hyperbolic group.*

I. Kapovich [21] worked on mapping-tori of injective free group endomorphisms, trying to avoid the assumption of malnormality of the endomorphism's image. We consider the group given by its standard presentation of mapping-torus group. Our proof relies on an approximation of the geodesics in the Cayley complex of the group for this presentation. Let α be an automorphism of F_n. Let G_α be the mapping-torus group of (α, F_n). The above Cayley complex for G_α has a very particular structure. It carries a non-singular semi-flow and this semi-flow is transverse to a foliation of the complex by trees. A non-singular semi-flow is a one-parameter family $(\sigma_t)_{t \in \mathbf{R}^+}$ of continuous maps of the 2-complex, depending continuously on the parameter and satisfying the usual properties of a flow: $\sigma_0 = \mathrm{Id}$, $\sigma_{t+t'} = \sigma_t \circ \sigma_{t'}$.

Let Γ be a graph with fundamental group F_n. Let $\psi \colon \Gamma \to \Gamma$ be a simplicial map on Γ which induces α on the fundamental group of Γ. Let $K = (\Gamma \times [0,1])/((x,1) \sim (\psi(x), 0))$ be the mapping-torus of (ψ, Γ). Then K is a simple example of a 2-complex equipped with a non-singular semi-flow. The orbits of the semi-flow are the concatenation of intervals $\{x\} \times [0,1]$, $x \in \Gamma$, glued together by identifying $(x,1)$ with $(\psi(x), 0)$. Moreover the 2-complex is foliated with compact graphs $\Gamma \times \{t\}$ transverse to the semi-flow. The universal covering of this 2-complex is the Cayley complex of G_α for the standard presentation as a mapping-torus group. Let us describe this universal covering. The universal covering of Γ is a tree T. Let $\widetilde{\psi} \colon T \to T$ be a simplicial lift of ψ. That is, if $\pi \colon \Gamma \to T$ is the covering-map, $\psi \circ \pi = \pi \circ \widetilde{\psi}$. Since ψ induces an automorphism on $\pi_1(\Gamma)$, the universal covering of K is homeomorphic to the quotient of $\bigsqcup_{n \in \mathbf{Z}} T \times [n, n+1]$ by the identification of

$(x,n+1) \in T\times[n,n+1]$ with $(\widetilde{\psi}(x),n+1) \in T\times[n+1,n+2]$. Such a topological space is called the *mapping-telescope of* $(\widetilde{\psi},T)$. As a corollary of our main theorem we obtain an analogue for mapping-telescopes of Thurston's theorem for mapping-tori of surface homeomorphisms. The structure of graph or of 2-complex which exists when dealing, as above, with Cayley complexes of mapping-torus groups is irrelevant. We only require that T be a 0-hyperbolic metric space, that is a geodesic metric space whose geodesic triangles are tripods. Equivalently, such a T is an $\mathbf{R}$*-tree*. We refer the reader to [2] or [8] for the equivalence of these two notions and to [2] for a survey about $\mathbf{R}$-trees. Let us observe that Bowditch [4] refers, without further proof, to [3] to state a theorem about the Gromov-hyperbolicity of mapping-telescopes of $\mathbf{R}$-graphs. A weak version of our result gives a complete proof of such a result in the case of $\mathbf{R}$-trees:

THEOREM 0.2. *Let (T,d_T) be an $\mathbf{R}$-tree. Let $\widetilde{\psi}\colon T \to T$ be a continuous map on T which satisfies the following properties:*

1) *There exist $\mu \geq 1$ and $K \geq 0$ such that $\mu d_T(x,y) \geq d_T(\widetilde{\psi}(x),\widetilde{\psi}(y)) \geq \frac{1}{\mu}d_T(x,y) - K$.*
2) *There exist $\lambda > 1$, $N \geq 1$ and $M \geq 0$ such that for any pair of points x, y in T with $d_T(x,y) \geq M$, either $d_T(\widetilde{\psi}^N(x),\widetilde{\psi}^N(y)) \geq \lambda\, d_T(x,y)$ or $d_T(x_N,y_N) \geq \lambda\, d_T(x,y)$ for some x_N, y_N with $\widetilde{\psi}^N(x_N) = x$, $\widetilde{\psi}^N(y_N) = y$.*

Then the mapping-telescope of $(\widetilde{\psi},T)$ is a Gromov-hyperbolic metric space for some mapping-telescope metric.

Let us briefly explain what a mapping-telescope metric is. Roughly speaking, at each point in the mapping-telescope we can move in two directions: along a leaf $T \times \{t\}$, or along a path which is a concatenation of intervals $\{x\} \times [n,n+1]$, $x \in T$. The lengths in the vertical direction are measured using the obvious parametrization. We provide the trees $T \times \{t\}$ with a metric. Then the mapping-telescope metric is defined as follows: the distance between two points x, y is the shortest path from x to y among all paths obtained as sequences of horizontal and vertical moves.

We deal with more general spaces than mapping-telescopes. The reader will find in Section 4 the precise statement of our result. The spaces under consideration are called forest-stacks. We only need on the one hand the existence of a non-singular semi-flow and, on the other hand, the existence of a transverse foliation by forests. We allow the homeomorphism-types of the forests to vary along $\mathbf{R}$. We refer the reader to Remark 13.8 for a brief discussion about direct applications of our main theorem, which we chose not to develop here for the sake of a clearer and shorter presentation.

In Section 1, we give an illustration, and a proof, of our theorem in a very particular case. Although very simple, the basic ideas of the sequel appear here. Sections 2 to 11 form the heart of the paper. In Sections 2 and 3 we define the objects under study. In Section 4 we state our theorem about forest-stacks. The statements of the other results, concerning mapping-telescopes and mapping-torus groups, appear in Sections 12 and 13. After some preliminary work (Section 5), we study the so-called straight quasi geodesics in forest-stacks equipped with strongly hyperbolic semi-flows (Sections 6 and 7). We rely upon these last two sections to give an approximation of straight quasi geodesics in fine position with respect to a horizontal one (Section 8), and then in Section 9 to show how to put a straight quasi geodesic in fine position with respect to a horizontal one. In Section 10 we gather all these results to prove that straight quasi geodesic bigons are thin. We conclude in Section 11. Building on this work, we give in [13] a generalization of the Bestvina-Feighn theorem in the 'relative hyperbolicity' context.

ACKNOWLEDGEMENTS. The author acknowledges support from the Swiss National Science Foundation. Warm thanks are also due to P. de la Harpe, M. Lustig, T. Vust and, last but not least, to I. Kapovich who helped greatly to simplify and make clearer the paper, in particular by suggesting Lemma 11.1.

Since they play the central role in this paper, we briefly specify what we mean by *Gromov hyperbolic metric spaces.* Gromov introduced the notion of (r,s)-*quasi geodesic space* in [19]: A metric space (X,d) is an (r,s)-quasi geodesic space if, for any two points x, y in X there is an (r,s)-*chain*, i.e. a finite set of points $x = x_0, x_1, \dots, x_k = y$ such that $d(x_{i-1},x_i) \leq r$ for $i = 1,\dots,k$ and that $\sum_{i=1}^{k} d(x_{i-1},x_i) \leq sd(x,y)$. A quasi geodesic metric space is a metric space which is (r,s)-quasi geodesic for some non negative real constants r, s. An (r,s)-chain triangle in a quasi geodesic metric space is a triangle whose sides are (r,s)-chains. A chain triangle is δ-thin, $\delta \geq 0$, if any side is in the δ-neighborhood of the union of the other two sides. We say that chain triangles in an (r,s)-quasi geodesic metric space X are thin if there exists a $\delta \geq 0$ such that any (r,s)-chain triangle in X is δ-thin. In this case, X is a Gromov-hyperbolic metric space; more precisely, X is a δ-hyperbolic metric space. In the entire paper, unless otherwise specified, '(quasi) geodesic(s)' means 'finite length (quasi) geodesic(s)'.