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## APPENDIX: EXAMPLES OF HOLOMORPHIC ENDOMORPHISMS

The following examples appeared after the discussion I had with Spencer Bloch and David Mumford.

## (1) TWISTED HOPF MANIFOLDS

Divide  $\mathbf{C}^n \setminus 0$  by the action of a linear operator  $A$  without eigenvalues in the unit disk. All endomorphisms of the quotient ( $n > 1$ ) come from polynomial maps  $\tilde{f}: \mathbf{C}^n \rightarrow \mathbf{C}^n$ . For such an endomorphism, its entropy and "lov" are probably equal to "log deg".

EXAMPLE.  $A: (z_1, z_2) \mapsto (\lambda_1 z_1, \lambda_2 z_2)$  and  $\tilde{f}: (z_1, z_2) \mapsto (z_1^p, z_2^p)$ .

## (2) GENERALIZED HOPF MANIFOLDS

Let  $f_0: X_0 \rightarrow X_0$  be an endomorphism. Take a line bundle  $L$  over  $X_0$  such that  $f_0^*(L) = \bigotimes^p L$  ( $:= L \otimes \cdots \otimes L$ ,  $p$  times). Locating such an  $L$  is usually quite easy by looking at  $\text{Pic}(X_0)$ . Denote by  $Y$  the total space of  $L$ . There is a fiberwise map  $\tilde{f}: Y \rightarrow Y$  lifting  $f_0$  and acting on fibers as  $z \mapsto z^p$ . If we divide  $Y$  by a fiberwise action of  $\mathbf{Z}$  (it is  $z \mapsto z_0 z$ ,  $z_0 \neq 0$ , in each fiber) we get  $f: Y/\mathbf{Z} \rightarrow Y/\mathbf{Z}$ .

There is another way to compactify  $Y$  by taking the total space of the one-dimensional projective bundle associated to  $L$ . The endomorphism  $\tilde{f}$  canonically extends to this compactification.

## (3) THE CALABI-ECKMANN MANIFOLDS

Let us take  $(\mathbf{C}^k \times \mathbf{C}^\ell) \setminus ((\mathbf{C}^k \times 0) \cup (0 \times \mathbf{C}^\ell))$  and divide by the following action of  $\mathbf{C}$ :

$$(z_1, z_2) \mapsto (A_1^\lambda z_1, A_2^\lambda z_2), \quad \lambda \in \mathbf{C}.$$

$A_1$  and  $A_2$  are appropriate linear operators in  $\mathbf{C}^k$  and  $\mathbf{C}^\ell$ . For example,  $A_1^\lambda = \exp \lambda$ ,  $A_2^\lambda = \exp i\lambda$ , where  $\lambda$  is a scalar. In the last case, the factor manifold possesses an endomorphism  $f$  which lifts to the following polynomial map

$$\mathbf{C}^k \times \mathbf{C}^\ell \rightarrow \mathbf{C}^k \times \mathbf{C}^\ell : (z_1, \dots, z_{k+\ell}) \mapsto (z_1^p, \dots, z_{k+\ell}^p).$$

Recall that the Calabi-Eckmann manifolds are diffeomorphic to  $\mathbf{S}^{2k-1} \times \mathbf{S}^{2\ell-1}$ . The above map  $f$  has degree  $d = (2(k + \ell - 1))^p$  and  $h(f) = \text{lov } f = \log d$ .

## (4) BLOWING UP

Let us take  $W \subset V_0$  and an endomorphism  $f: V_0 \rightarrow V_0$  such that  $f^{-1}(W) = W$ . The endomorphism  $f$  can be sometimes lifted to the manifold  $V$  obtained by blowing up  $W$ .

EXAMPLE.  $V_0 = \mathbf{CP}^1 \times \mathbf{CP}^1$ ,  $W$  is the single point  $(0, 0)$ , and  $f: (z_1, z_2) \mapsto (z_1^p, z_2^p)$ .

## (5) CONCLUDING REMARKS

A typical compact complex manifold has very few endomorphisms. For example, manifolds with nontrivial Kobayashi volume have no endomorphisms of degree  $\geq 2$ . Do Grassmann manifolds have such endomorphisms? (No, see [3'].)

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