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## APPENDIX: EXAMPLES OF HOLOMORPHIC ENDOMORPHISMS

The following examples appeared after the discussion I had with Spencer Bloch and David Mumford.

### (1) TWISTED HOPF MANIFOLDS

Divide $\mathbf{C}^n \setminus 0$ by the action of a linear operator $A$ without eigenvalues in the unit disk. All endomorphisms of the quotient ($n > 1$) come from polynomial maps $\widetilde{f}\colon \mathbf{C}^n \to \mathbf{C}^n$. For such an endomorphism, its entropy and "lov" are probably equal to "log deg".

EXAMPLE. $A\colon (z_1, z_2) \longmapsto (\lambda_1 z_1, \lambda_2 z_2)$ and $\widetilde{f}\colon (z_1, z_2) \longmapsto (z_1^p, z_2^p)$.

### (2) GENERALIZED HOPF MANIFOLDS

Let $f_0\colon X_0 \to X_0$ be an endomorphism. Take a line bundle $L$ over $X_0$ such that $f_0^*(L) = \bigotimes^p L$ $(:= L \otimes \cdots \otimes L,\ \ p$ times$)$. Locating such an $L$ is usually quite easy by looking at $\mathrm{Pic}(X_0)$. Denote by $Y$ the total space of $L$. There is a fiberwise map $\widetilde{f}\colon Y \to Y$ lifting $f_0$ and acting on fibers as $z \longmapsto z^p$. If we divide $Y$ by a fiberwise action of $\mathbf{Z}$ (it is $z \longmapsto z_0 z$, $z_0 \neq 0$, in each fiber) we get $f\colon Y/\mathbf{Z} \to Y/\mathbf{Z}$.

There is another way to compactify $Y$ by taking the total space of the one-dimensional projective bundle associated to $L$. The endomorphism $\widetilde{f}$ canonically extends to this compactification.

### (3) THE CALABI-ECKMANN MANIFOLDS

Let us take $\left(\mathbf{C}^k \times \mathbf{C}^\ell\right) \setminus \left((\mathbf{C}^k \times 0) \cup (0 \times \mathbf{C}^\ell)\right)$ and divide by the following action of $\mathbf{C}$:

$$(z_1, z_2) \longmapsto \left(A_1^\lambda z_1, A_2^\lambda z_2\right), \qquad \lambda \in \mathbf{C}.$$

$A_1$ and $A_2$ are appropriate linear operators in $\mathbf{C}^k$ and $\mathbf{C}^\ell$. For example, $A_1^\lambda = \exp \lambda$, $A_2^\lambda = \exp i\lambda$, where $\lambda$ is a scalar. In the last case, the factor manifold possesses an endomorphism $f$ which lifts to the following polynomial map

$$\mathbf{C}^k \times \mathbf{C}^\ell \to \mathbf{C}^k \times \mathbf{C}^\ell \ :\ (z_1, \ldots, z_{k+\ell}) \longmapsto (z_1^p, \ldots, z_{k+\ell}^p).$$

Recall that the Calabi-Eckmann manifolds are diffeomorphic to $\mathbf{S}^{2k-1} \times \mathbf{S}^{2\ell-1}$. The above map $f$ has degree $d = (2(k+\ell-1))^p$ and $h(f) = \mathrm{lov}\, f = \log d$.

(4) BLOWING UP

Let us take $W \subset V_0$ and an endomorphism $f \colon V_0 \to V_0$ such that $f^{-1}(W) = W$. The endomorphism $f$ can be sometimes lifted to the manifold $V$ obtained by blowing up $W$.

EXAMPLE. $V_0 = \mathbf{C}P^1 \times \mathbf{C}P^1$, $W$ is the single point $(0,0)$, and $f \colon (z_1, z_2) \mapsto (z_1^p, z_2^p)$.

(5) CONCLUDING REMARKS

A typical compact complex manifold has very few endomorphisms. For example, manifolds with nontrivial Kobayashi volume have no endomorphisms of degree $\geq 2$. Do Grassmann manifolds have such endomorphisms? (No, see [3′].)

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