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n-dimensional Hausdorff measure with respect to the Riemann product metric in X^k. Set

$$\operatorname{lov} \Gamma = \limsup_{k\to\infty} \frac{1}{k} \log \operatorname{Vol} \Gamma_k .$$

For an f we set $\operatorname{lov} f = \operatorname{lov} \Gamma_f$. This is a smooth invariant of f (it does not depend on the choice of the Riemann metric).

Our invariant "lov" is sometimes more accessible than entropy and for a holomorphic f we are going to prove that

$$h(f) \leq \operatorname{lov} f . \tag{1.0}$$

DENSITY

Denote by $\operatorname{Dens}_\epsilon(\Gamma_k, \gamma)$, for $\gamma \in \Gamma_k \subset X^k$, the n-dimensional measure of the intersection of Γ_k with the ball (in the Riemannian product metric) of radius ϵ centered at γ. Set $\operatorname{Dens}_\epsilon(\Gamma_k) = \inf_{k\in\Gamma_k} \operatorname{Dens}_\epsilon(\Gamma_k, \gamma)$, and then $\operatorname{lodn}_\epsilon \Gamma = \liminf_{k\to\infty} \frac{1}{k} \log \operatorname{Dens}_\epsilon \Gamma_k$, and finally

$$\operatorname{lodn} \Gamma = \lim_{\epsilon\to 0} \operatorname{lodn}_\epsilon \Gamma .$$

Observe that $\operatorname{Vol} \geq \operatorname{Cap}_{2\epsilon} \operatorname{Dens}_\epsilon$ and hence that

$$h(V) \leq \operatorname{lov} \Gamma - \operatorname{lodn} \Gamma . \tag{1.1}$$

§2. ESTIMATES OF DENSITY

Consider a Riemannian manifold W (it will be X^k in the sequel) and an n-dimensional subvariety $V \subset W$. We suppose that the boundary of V (if there is such) does not intersect the ball $B_\epsilon \subset W$ of radius $\epsilon > 0$ centered at a point $v_0 \in V$. We suppose also that the injectivity radius of W at v_0 is at least ϵ, i.e. the exponential map $T_{v_0}(W) \to W$ is injective in the ϵ-ball in $T_{v_0}(W)$.

DENSITY OF A MINIMAL VARIETY

If the sectional curvature in B_ϵ is not greater than K and V is minimal, then

$$\operatorname{Vol}(V \cap B_\epsilon) \geq C > 0 , \tag{2.0}$$

where the constant C depends on n, K, and ϵ, but does not depend on $\dim W$.

The proof is given below. This fact is well known and C is equal to the volume of the ϵ-ball in the n-dimensional space of constant curvature K. Our application of (2.0) to complex geometry is based on

FEDERER'S THEOREM. *Analytic subvarieties of a Kähler manifold are minimal.*

Thus we can apply (2.0), conclude that $\operatorname{lodn}\Gamma = 0$ and obtain (1.0) in the Kähler case by using (1.1).

Proof of (2.0). We restrict ourselves to the case when W is the Euclidean space and V is nonsingular. Denote by A_ϵ the $(n-1)$-dimensional volume of the boundary $V \cap \partial B_\epsilon$.

Minimality of V implies

$$V_\epsilon \geq \operatorname{Vol} \operatorname{Co}_\epsilon = \frac{\epsilon}{n} A_\epsilon \,, \tag{2.1}$$

where $\operatorname{Co}_\epsilon$ is the cone over A_ϵ centered at v_0.

On the other hand

$$V_\epsilon \geq \int_0^\epsilon A_\tau \, d\tau \,. \tag{2.2}$$

Regularity of V implies that

$$\lim_{\epsilon \to 0} \frac{V_\epsilon}{\epsilon^n} = \lim_{\epsilon \to 0} \frac{A_\epsilon}{n\epsilon^{n-1}} = C_n \,, \tag{2.3}$$

where C_n is the volume of the unit ball in $\mathbf{R}^n$.

Combining (2.1), (2.2) and (2.3), we get

$$V_\epsilon \geq C_n \epsilon^n \,, \tag{2.4}$$

which implies (2.0) in the Euclidean case.

Proof of (1.0). As we mentioned above, (2.4) implies (1.0), but only in the Euclidean case where (2.4) is proven. But the local nature of the density enables us to reduce the general case to the Euclidean one: near each point $x \in X$ we equip the complex manifold X (we suppose that X is compact without boundary) with a flat (i.e. Euclidean) Kähler structure and use the product structure near each point from X^k. Independence of "lodn" upon the choice of the metric allows one to apply (2.4) to derive the vanishing of "lodn" and thus the desired inequality $h \leq \operatorname{lov}$.