

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 49 (2003)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON THE ENTROPY OF HOLOMORPHIC MAPS
Autor: GROMOV, Mikhaïl
Kapitel: §2. ESTIMATES OF DENSITY
DOI: <https://doi.org/10.5169/seals-66687>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 03.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

n -dimensional Hausdorff measure with respect to the Riemann product metric in X^k . Set

$$\text{lov } \Gamma = \limsup_{k \rightarrow \infty} \frac{1}{k} \log \text{Vol } \Gamma_k.$$

For an f we set $\text{lov } f = \text{lov } \Gamma_f$. This is a smooth invariant of f (it does not depend on the choice of the Riemann metric).

Our invariant "lov" is sometimes more accessible than entropy and for a holomorphic f we are going to prove that

$$(1.0) \quad h(f) \leq \text{lov } f.$$

DENSITY

Denote by $\text{Dens}_\epsilon(\Gamma_k, \gamma)$, for $\gamma \in \Gamma_k \subset X^k$, the n -dimensional measure of the intersection of Γ_k with the ball (in the Riemannian product metric) of radius ϵ centered at γ . Set $\text{Dens}_\epsilon(\Gamma_k) = \inf_{\gamma \in \Gamma_k} \text{Dens}_\epsilon(\Gamma_k, \gamma)$, and then $\text{lodn}_\epsilon \Gamma = \liminf_{k \rightarrow \infty} \frac{1}{k} \log \text{Dens}_\epsilon \Gamma_k$, and finally

$$\text{lodn } \Gamma = \lim_{\epsilon \rightarrow 0} \text{lodn}_\epsilon \Gamma.$$

Observe that $\text{Vol} \geq \text{Cap}_{2\epsilon} \text{Dens}_\epsilon$ and hence that

$$(1.1) \quad h(V) \leq \text{lov } \Gamma - \text{lodn } \Gamma.$$

§2. ESTIMATES OF DENSITY

Consider a Riemannian manifold W (it will be X^k in the sequel) and an n -dimensional subvariety $V \subset W$. We suppose that the boundary of V (if there is such) does not intersect the ball $B_\epsilon \subset W$ of radius $\epsilon > 0$ centered at a point $v_0 \in V$. We suppose also that the injectivity radius of W at v_0 is at least ϵ , i.e. the exponential map $T_{v_0}(W) \rightarrow W$ is injective in the ϵ -ball in $T_{v_0}(W)$.

DENSITY OF A MINIMAL VARIETY

If the sectional curvature in B_ϵ is not greater than K and V is minimal, then

$$(2.0) \quad \text{Vol}(V \cap B_\epsilon) \geq C > 0,$$

where the constant C depends on n , K , and ϵ , but does not depend on $\dim W$.

The proof is given below. This fact is well known and C is equal to the volume of the ϵ -ball in the n -dimensional space of constant curvature K . Our application of (2.0) to complex geometry is based on

FEDERER'S THEOREM. *Analytic subvarieties of a Kähler manifold are minimal.*

Thus we can apply (2.0), conclude that $\text{lodn } \Gamma = 0$ and obtain (1.0) in the Kähler case by using (1.1).

Proof of (2.0). We restrict ourselves to the case when W is the Euclidean space and V is nonsingular. Denote by A_ϵ the $(n-1)$ -dimensional volume of the boundary $V \cap \partial B_\epsilon$.

Minimality of V implies

$$(2.1) \quad V_\epsilon \geq \text{Vol Co}_\epsilon = \frac{\epsilon}{n} A_\epsilon,$$

where Co_ϵ is the cone over A_ϵ centered at v_0 .

On the other hand

$$(2.2) \quad V_\epsilon \geq \int_0^\epsilon A_\tau d\tau.$$

Regularity of V implies that

$$(2.3) \quad \lim_{\epsilon \rightarrow 0} \frac{V_\epsilon}{\epsilon^n} = \lim_{\epsilon \rightarrow 0} \frac{A_\epsilon}{n\epsilon^{n-1}} = C_n,$$

where C_n is the volume of the unit ball in \mathbf{R}^n .

Combining (2.1), (2.2) and (2.3), we get

$$(2.4) \quad V_\epsilon \geq C_n \epsilon^n,$$

which implies (2.0) in the Euclidean case.

Proof of (1.0). As we mentioned above, (2.4) implies (1.0), but only in the Euclidean case where (2.4) is proven. But the local nature of the density enables us to reduce the general case to the Euclidean one: near each point $x \in X$ we equip the complex manifold X (we suppose that X is compact without boundary) with a flat (i.e. Euclidean) Kähler structure and use the product structure near each point from X^k . Independence of "lodn" upon the choice of the metric allows one to apply (2.4) to derive the vanishing of "lodn" and thus the desired inequality $h \leq \text{lov}$.