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me to write this paper. I also appreciate the hospitality of IHES, which made possible my involvement in this story. I am especially thankful to Dennis Sullivan, who took pains to read the paper and to clean it up of multiple errors.

STRUCTURE OF THE PAPER. We start with a geometric outlook on topological entropy and reduce our inequality (0.1) to standard facts about minimal varieties. We discuss next the real algebraic analogue of (0.1) and a generalization to maps. We conclude with an estimate of the entropy involving the mean curvature.

### §1. NOTATION AND DEFINITIONS

For a space  $X$  we denote by  $X^k$  the product  $X \times X \times \dots \times X$  ( $k$  factors). A *graph*  $\Gamma$  over  $X$  is by definition an arbitrary set  $\Gamma \subset X^2$ . When  $X$  is finite this is the usual definition of an oriented graph (with loops). The graph of a map  $X \rightarrow X$  gives another example.

For a graph  $\Gamma$  we denote by  $\Gamma_k \subset X^k$  the set of strings  $(x_1, \dots, x_i, \dots, x_k)$ ,  $x_i \in X$ , where each pair  $(x_{i-1}, x_i) \in X^2$  is contained in  $\Gamma$ .

When  $X$  is endowed with a metric, we call  $\epsilon$ -cubes products in  $X^k$  of balls from  $X$  of radius  $\epsilon$ . For a set  $Y \subset X^k$  we denote by  $\text{Cap}_\epsilon Y$  the minimal number of  $\epsilon$ -cubes needed to cover  $Y$ .

#### ENTROPY

Set  $h_\epsilon(\Gamma) = \limsup_{k \rightarrow \infty} \frac{1}{k} \log \text{Cap}_\epsilon \Gamma_k$ , and  $h(\Gamma) = \lim_{\epsilon \rightarrow 0} h_\epsilon(\Gamma)$ , for  $\Gamma \subset X^2$ .

When  $f$  is an endomorphism  $X \rightarrow X$ , we define its *entropy*  $h(f)$  as the entropy of its graph  $\Gamma_f$ . If the space  $X$  is compact, the definition does not depend on the choice of the metric [2]. Observe that the entropy of a general graph  $\Gamma$  is equal to the entropy of the shift in  $\Gamma_\infty \subset X^\infty$ :  $\Gamma_\infty$  is the space of doubly infinite strings  $(x_i)_{i=\dots, -1, 0, 1, \dots}$  with the product topology, and the shift maps  $(x_i)$  to  $(x_{i+1})$ . For finite  $X$ , we come to the usual definition of the Markov shift.

#### VOLUME

From now on,  $X$  is a Riemannian manifold and  $n = \dim X$ ,  $\Gamma \subset X^2$ . We denote by  $\text{Vol} \Gamma_k$  the  $n$ -dimensional volume of  $\Gamma_k \subset X^k$ , i.e. the

$n$ -dimensional Hausdorff measure with respect to the Riemann product metric in  $X^k$ . Set

$$\text{lov } \Gamma = \limsup_{k \rightarrow \infty} \frac{1}{k} \log \text{Vol } \Gamma_k.$$

For an  $f$  we set  $\text{lov } f = \text{lov } \Gamma_f$ . This is a smooth invariant of  $f$  (it does not depend on the choice of the Riemann metric).

Our invariant "lov" is sometimes more accessible than entropy and for a holomorphic  $f$  we are going to prove that

$$(1.0) \quad h(f) \leq \text{lov } f.$$

#### DENSITY

Denote by  $\text{Dens}_\epsilon(\Gamma_k, \gamma)$ , for  $\gamma \in \Gamma_k \subset X^k$ , the  $n$ -dimensional measure of the intersection of  $\Gamma_k$  with the ball (in the Riemannian product metric) of radius  $\epsilon$  centered at  $\gamma$ . Set  $\text{Dens}_\epsilon(\Gamma_k) = \inf_{\gamma \in \Gamma_k} \text{Dens}_\epsilon(\Gamma_k, \gamma)$ , and then  $\text{lodn}_\epsilon \Gamma = \liminf_{k \rightarrow \infty} \frac{1}{k} \log \text{Dens}_\epsilon \Gamma_k$ , and finally

$$\text{lodn } \Gamma = \lim_{\epsilon \rightarrow 0} \text{lodn}_\epsilon \Gamma.$$

Observe that  $\text{Vol} \geq \text{Cap}_{2\epsilon} \text{Dens}_\epsilon$  and hence that

$$(1.1) \quad h(V) \leq \text{lov } \Gamma - \text{lodn } \Gamma.$$

## §2. ESTIMATES OF DENSITY

Consider a Riemannian manifold  $W$  (it will be  $X^k$  in the sequel) and an  $n$ -dimensional subvariety  $V \subset W$ . We suppose that the boundary of  $V$  (if there is such) does not intersect the ball  $B_\epsilon \subset W$  of radius  $\epsilon > 0$  centered at a point  $v_0 \in V$ . We suppose also that the injectivity radius of  $W$  at  $v_0$  is at least  $\epsilon$ , i.e. the exponential map  $T_{v_0}(W) \rightarrow W$  is injective in the  $\epsilon$ -ball in  $T_{v_0}(W)$ .

#### DENSITY OF A MINIMAL VARIETY

*If the sectional curvature in  $B_\epsilon$  is not greater than  $K$  and  $V$  is minimal, then*

$$(2.0) \quad \text{Vol}(V \cap B_\epsilon) \geq C > 0,$$

*where the constant  $C$  depends on  $n$ ,  $K$ , and  $\epsilon$ , but does not depend on  $\dim W$ .*