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**Autor:** Tráng, Lê Dng / Seade, José / Verjovsky, Alberto  
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that we considered in Sections 2 and 3 above. Similarly, if  $G$  is  $\mathbf{Z}/2\mathbf{Z}$  acting on  $Q$  as the antipodal map, then the corresponding extension to  $P_{\mathbf{C}}^2$  is given by complex conjugation.

### 3. $P_{\mathbf{C}}^2$ AND THE 4-SPHERE $S^4$

The previous discussion, restricted to  $n = 2$  and compared to the cohomogeneity 1 isometric action of  $\mathrm{SO}(3, \mathbf{R})$  on  $S^4$  constructed in [HL], motivates an equivariant version of the Arnold-Kuiper-Massey theorem [Ar1, Ar2, Ku, Ma1], saying that  $P_{\mathbf{C}}^2$  modulo conjugation is the 4-sphere. In this section we give a new proof of this theorem. We construct an explicit algebraic map  $\Phi: P_{\mathbf{C}}^2 \rightarrow S^4$ , which is equivariant with respect to the cohomogeneity 1 isometric actions of  $\mathrm{SO}(3, \mathbf{R})$  on  $P_{\mathbf{C}}^2$  and  $S^4$  and induces a diffeomorphism  $P_{\mathbf{C}}^2/\text{conjugation} \cong S^4$ .

We start by recalling the  $\mathrm{SO}(3, \mathbf{R})$ -action on  $S^4$ , as explained by Hsiang and Lawson in [HL; Example 1.4].

Let  $\mathcal{S}$  be the vector space of real  $3 \times 3$ , traceless and symmetric matrices. As a real vector space  $\mathcal{S}$  is  $\mathbf{R}^5$ , and it can be equipped with a metric given by the inner product  $(A, B) \mapsto \text{trace}(AB)$ . Let  $\mathcal{S}^{(4)}$  be the space of matrices in  $\mathcal{S}$  with norm 1. One has an obvious diffeomorphism  $S^4 \cong \mathcal{S}^{(4)}$ , which becomes isometric if we endow  $S^4$  with its usual round metric and  $\mathcal{S}^{(4)}$  with the metric given by the inner product in  $\mathcal{S}$ . We shall identify these two spaces in the sequel, denoting both of them by  $S^4$  indistinctly. The group  $\mathrm{SO}(3, \mathbf{R})$  acts on  $\mathcal{S}$  by  $A \mapsto O^t A O$ , where  $O^t$  is the transposed matrix (which is equal, in our case, to  $O^{-1}$ ). This induces an isometric action  $\Gamma$  of  $\mathrm{SO}(3, \mathbf{R})$  on  $S^4$ . This action on  $S^4$  has two disjoint copies of  $P_{\mathbf{R}}^2$  as special fibres (see the remark at the end of this section). The space of orbits is the interval  $[0, 1]$ , with the endpoints giving the special orbits. Each principal orbit (i.e. the orbits of highest dimension) is a flag manifold

$$F^3(2, 1) \cong \mathrm{SO}(3, \mathbf{R}) / (\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}) \cong L(4, 1) / (\mathbf{Z}/2\mathbf{Z}),$$

of pairs  $(P, l)$  with  $P$  a plane in  $\mathbf{R}^3$  and  $l$  line in  $P$ , where  $L(4, 1)$  is the lens space  $S^3 / (\mathbf{Z}/4\mathbf{Z}) \cong \mathrm{SO}(3, \mathbf{R}) / (\mathbf{Z}/2\mathbf{Z})$ .

Let us give a similar description of  $P_{\mathbf{C}}^2$ . Let

$$\mathfrak{H}(3, \mathbf{C}) = \{H \in M(3, \mathbf{C}) \mid H = H^*\}$$

be the space of complex  $3 \times 3$  Hermitian matrices, where  $H^* = \bar{H}^t$  is the adjoint matrix of  $H$ , obtained by first conjugating each entry of  $H$  and then

transposing the matrix. We equip  $\mathfrak{H}(3, \mathbb{C})$  with the Hermitian inner product

$$(3.1) \quad \langle H_1, H_2 \rangle = \frac{1}{2} \operatorname{trace}(H_1 H_2).$$

As a vector space, with this inner product,  $\mathfrak{H}(3, \mathbb{C})$  is the ordinary Euclidean space  $\mathbb{E}^9$ . Consider the subset  $P(2)$  of  $\mathfrak{H}(3, \mathbb{C})$  defined by

$$(3.2) \quad P(2) = \{H \in \mathfrak{H}(3, \mathbb{C}) \mid H^2 = H \text{ and } \operatorname{trace}(H) = 1\}.$$

LEMMA 3.3. *The set  $P(2)$  is a manifold, diffeomorphic to  $P_{\mathbb{C}}^2$ . Moreover, if we endow  $P(2)$  with the metric defined by (3.1), then  $P(2)$  is isometric to  $P_{\mathbb{C}}^2$  equipped with the Fubini-Study metric (of constant holomorphic sectional curvature 4).*

We remark that it is possible to describe  $P_{\mathbb{C}}^n$  in a similar way, but we restrict our attention to  $n = 2$  because this is all we need.

*Proof.* We claim that if  $H$  is in  $P(2)$ , then it is an orthogonal projection over a complex line. In fact, if  $H$  is in  $P(2)$ , then it is diagonalizable by a unitary matrix and its eigenvalues are 0 or 1, because  $H^2 = H$ . Since the trace is one, two eigenvalues must be 0 and the other is 1. Hence  $H$  is a surjection of  $\mathbb{C}^3$  over a complex line, and this map has to be an orthogonal projection because  $H$  is Hermitian. Conversely, it is clear that each line  $L \in \mathbb{C}^3$  determines a unique orthogonal projection of  $\mathbb{C}^3$ , and this is given by a matrix in  $P(2)$ . The diffeomorphism in Lemma 3.3 is achieved by the map that carries  $H$  into the corresponding line in  $\mathbb{C}^3$ . To prove that this map gives a metric equivalence, we notice that the unitary group  $U(3)$  acts on  $\mathfrak{H}(3, \mathbb{C})$  by  $H \mapsto U^* H U$ , and  $P(2)$  is an orbit of this action, with isotropy  $(U(2) \times U(1))$ . Thus,

$$P(2) \cong U(3)/(U(2) \times U(1)) \cong P_{\mathbb{C}}^2,$$

and the metric on  $P(2)$  is obviously  $U(3)$ -invariant. Hence the induced metric on  $P_{\mathbb{C}}^2$  is also  $U(3)$ -invariant, and this characterizes the Fubini-Study metric, up to scaling.  $\square$

We recall now that the quotient of  $P_{\mathbb{C}}^2$  by the complex conjugation  $j$  is a smooth manifold, which is not an obvious fact since  $j$  has fixed points. This is carefully explained in [Mar], so we only sketch a few ideas here. Away from the fixed point set  $\Pi \cong P_{\mathbb{R}}^2$ , the involution  $j$  is free, so the quotient is a smooth manifold. The problem is on  $\Pi$ . A tubular neighbourhood of

$\Pi$  in  $P_{\mathbb{C}}^2$  can be regarded as an open disk normal bundle, and conjugation carries each normal fibre into itself. Since the quotient of each normal 2-disk by the involution is again a 2-disk, it follows that the quotient  $P_{\mathbb{C}}^2/j$  is a topological manifold. Making this argument more carefully one gets that  $P_{\mathbb{C}}^2/j$  is in fact a  $PL$ -manifold, as noticed in [Ku], and therefore it is smooth, since every piecewise linear 4-manifold is smooth. In [Mar] Marin defines the smooth structure on  $P_{\mathbb{C}}^2/j$  directly, without using  $PL$ -structures. An important point is that the smooth structure on  $P_{\mathbb{C}}^2/j$  is such that the obvious projection  $P_{\mathbb{C}}^2 \rightarrow P_{\mathbb{C}}^2/j$  is differentiable.

Let us denote by  $\Gamma$  the aforementioned isometric action of  $SO(3, \mathbf{R})$  on  $S^4$ , and by  $\tilde{\Gamma}$  the standard action of  $SO(3, \mathbf{R})$  on  $P_{\mathbb{C}}^2$ , which is by isometries with respect to the Fubini-Study metric. This action is defined either by considering  $SO(3, \mathbf{R})$  as a subgroup of  $O(3, \mathbf{C})$ , acting on the space of lines in  $\mathbf{C}^3$ , or via the action of  $SO(3, \mathbf{R})$  on the space of matrices  $P(2) \subset H(3, \mathbf{C})$  given by

$$(O, A) \mapsto O^t A O.$$

By Lemma 3.3, both metrics on  $P_{\mathbb{C}}^2$  are equivalent; also for every  $O \in SO(3, \mathbf{R})$ ,  $H \in P(2)$  and  $v \in \mathbf{C}^3$  such that  $H(v) = v$ , one has  $O^t H O(O^{-1}(v)) = O^{-1}(v)$ , because  $O^{-1} = O^t$ . Hence both actions on  $P_{\mathbb{C}}^2 \cong P(2)$  are equivalent. Similarly, given the  $SO(3, \mathbf{R})$ -actions  $\tilde{\Gamma}$  on  $P_{\mathbb{C}}^2$  and  $\Gamma$  on  $S^4$ , we say that these actions are equivariant if there exists a map  $\Phi: P_{\mathbb{C}}^2 \rightarrow S^4$  which makes the following diagram commutative:

$$\begin{array}{ccc} SO(3, \mathbf{R}) \times P_{\mathbb{C}}^2 & \xrightarrow{\tilde{\Gamma}} & P_{\mathbb{C}}^2 \\ Id \times \Phi \downarrow & & \downarrow \Phi \\ SO(3, \mathbf{R}) \times S^4 & \xrightarrow{\Gamma} & S^4. \end{array}$$

In this case we say that  $\Phi$  *conjugates* the actions  $\Gamma$  and  $\tilde{\Gamma}$ . The map  $\Phi$  carries orbits into orbits, i.e. the decompositions of  $P_{\mathbb{C}}^2$  and  $S^4$  into orbits are (smoothly) equivalent.

Let us now state the equivariant Arnold-Kuiper-Massey theorem:

**THEOREM 3.4.** *There is a real algebraic equivariant map  $\Phi: P_{\mathbb{C}}^2 \rightarrow S^4$ , which is invariant by the complex conjugation  $j$  and induces a diffeomorphism  $P_{\mathbb{C}}^2/j \cong S^4$ , providing a conjugation between the isometric  $SO(3, \mathbf{R})$ -actions  $\tilde{\Gamma}$  on  $P_{\mathbb{C}}^2$  and  $\Gamma$  on  $S^4$ .*



We notice that Theorem 3.4, together with [HL], imply that the image of  $P_{\mathbf{R}}^2 \subset P_{\mathbf{C}}^2$  under the above map is the image of  $P_{\mathbf{R}}^2$  by the classical Veronese embedding  $(P_{\mathbf{C}}^2, P_{\mathbf{R}}^2) \hookrightarrow (P_{\mathbf{C}}^5, S^4)$ .

The proof of Theorem 3.4 follows from several lemmas below.

LEMMA 3.5. *Let  $A$  be a real  $(3 \times 3)$ -matrix. Then  $A$  is the real part of a matrix  $H$  in  $P(2)$  if and only if*

- i)  $A$  is symmetric with trace 1;
- ii)  $A$  has 0 as an eigenvalue and the other two eigenvalues  $\lambda_i$  and  $\lambda_j$  are roots of an equation of the form:

$$\lambda^2 - \lambda + k = 0,$$

for some constant  $k \in \mathbf{R}$  with  $0 \leq k \leq \frac{1}{4}$ .

If  $A$  and  $H$  are as above, and if  $O \in \text{SO}(3, \mathbf{R})$  is such that  $O^t A O$  is a diagonal matrix, then the imaginary part  $B$  of  $H$ , taken into its canonical form  $O^t B O$ , has only two possible non-zero entries, which are  $\pm\sqrt{k}$ . In particular, if  $k = 0$ , then  $H = A$ .

*Proof.* Let us consider a matrix  $H \in P(2)$  and decompose it into its real and imaginary parts:  $H = A + iB$ . Then one has  $\bar{H}^t = A^t - iB^t$ . Also  $H = \bar{H}^t$  because  $H$  is Hermitian. Hence  $A = A^t$  and  $B = -B^t$ , i.e.  $A$  is symmetric and  $B$  is anti-symmetric. Thus the trace of  $A$  is 1, proving statement (i). One also has

$$H^2 = A^2 - B^2 + i(AB + BA),$$

and  $H^2 = H$  because  $H$  is in  $P(2)$ . Therefore  $A = A^2 - B^2$  and  $B = AB + BA$ .

Now,  $A$  is symmetric, and so is  $A^2$ ; these two matrices obviously commute, so they can be diagonalized simultaneously by a matrix  $O \in \text{SO}(3, \mathbf{R})$ . Since  $B^2 = A^2 - A$ , one knows that  $O^t B^2 O$  is also diagonal:

$$O^t B^2 O = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix},$$

with  $\mu_i = \lambda_i^2 - \lambda_i$ , for each  $i = 1, 2, 3$ , where the  $\lambda_i$  are the eigenvalues of  $A$ . But  $B$  is antisymmetric and commutes with  $B^2$ , which is symmetric. Hence the same matrix  $O$  takes  $B$  to its canonical form:

$$O^t B O = \begin{pmatrix} 0 & a & c \\ -a & 0 & b \\ -c & -b & 0 \end{pmatrix}$$

for some  $a, b, c \in \mathbf{C}$ . This implies that

$$O^t B^2 O = (O^t B O)(O^t B O) = \begin{pmatrix} -a^2 - c^2 & -bc & ab \\ -bc & -a^2 - b^2 & -ac \\ ab & -ac & -b^2 - c^2 \end{pmatrix},$$

which we know is a diagonal matrix. Therefore two of the numbers  $a, b, c$  must be zero. Assume for instance that  $a$  and  $b$  are 0, then both eigenvalues  $\lambda_1$  and  $\lambda_3$  are roots of the polynomial

$$\lambda^2 - \lambda + c^2 = 0.$$

This implies that

$$\lambda_1 + \lambda_3 = 1 \quad \text{and} \quad \lambda_1 \cdot \lambda_3 = c^2 \geq 0.$$

Hence  $\lambda_2 = 0$  (because the trace of  $A$  is 1), so 0 is an eigenvalue of  $A$ . The other eigenvalues  $\lambda_1$  and  $\lambda_3$  must both be  $\geq 0$  and  $\leq 1$ , because their product is non-negative and their sum is 1. Moreover the roots must be real, therefore  $k = c^2 \leq \frac{1}{4}$ , proving statement (ii).

Also, in this case the eigenvalues of  $A$  determine the imaginary part  $B$  of  $H$  up to sign:

$$B = \pm O \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ -c & 0 & 0 \end{pmatrix} O^t,$$

with  $c^2 = \lambda_1 - \lambda_3^2 = \lambda_3 - \lambda_1^2$ , proving in this case the last statement of Lemma 3.5. The other cases, when either  $a = c = 0$  or  $b = c = 0$ , are similar to the previous one. This proves that if  $A = \Re(H)$  for some matrix  $H \in P(2)$ , then  $A$  is as stated in Lemma 3.5. Conversely, given  $A$  satisfying these conditions, the above arguments tell us how to construct  $B$  so that these matrices are the real and imaginary parts of some  $H$  in  $P(2)$ .  $\square$

Now, given  $H \in P(2)$ , its real part is  $\Re(H) = \frac{1}{2}(H + \bar{H})$ . Define

$$\psi: P(2) \rightarrow M(3, \mathbf{R}),$$

the space  $M(3, \mathbf{R})$  being the space of real  $(3 \times 3)$ -matrices, by the formula

$$(3.6) \quad \psi(H) = \frac{1}{3} I_3 - \Re(H) \in M(3, \mathbf{R}),$$

where  $I_3$  is the  $(3 \times 3)$ -identity matrix. In other words,  $\psi(H)$  is the real part of the matrix  $(\frac{1}{3} I_3 - H)$ . Since  $H \in P(2)$ , it follows that  $\psi(H)$  is actually contained in  $\mathcal{S}$ .

It is clear that the above action of  $\text{SO}(3, \mathbf{R})$  on  $P(2)$  given by conjugation is equivalent, via the above diffeomorphism  $P(2) \cong P_{\mathbf{C}}^2$ , with the standard action

studied in §2 and §3 above. It is also clear that, for every  $O \in \text{SO}(3, \mathbf{R})$ , one has

$$\psi(O^t H O) = \frac{1}{3}I - \frac{1}{2}(O^t(H + \bar{H})O) = O^t\left(\frac{1}{3}I - \frac{1}{2}(H + \bar{H})\right)O = O^t\psi(H)O.$$

Hence we have

LEMMA 3.7. *The map  $\psi$  is equivariant. That is, for every  $O \in \text{SO}(3, \mathbf{R})$  and  $H \in P(2)$ , one has  $\psi(O^t H O) = O^t\psi(H)O$ .*

LEMMA 3.8. *Given  $S \in \mathcal{S} - \{0\}$ , there exists a unique positive  $t \in \mathbf{R}$ , such that the matrix  $(\frac{1}{3}I - tS)$  is the real part of some matrix  $H \in P(2)$ .*

*Proof.* By Lemma 3.7, we may assume that  $S$  is diagonal. Hence the matrix  $\hat{S}_t = (\frac{1}{3}I - tS)$  is also diagonal, say

$$\hat{S}_t = \begin{pmatrix} \lambda_1(t) & 0 & 0 \\ 0 & \lambda_2(t) & 0 \\ 0 & 0 & \lambda_3(t) \end{pmatrix}$$

with  $\lambda_i(t) = \frac{1}{3} - t\mu_i$ , where the  $\mu_i$  are the eigenvalues of  $S$ . We notice that for all  $t \in \mathbf{R}$ , one has

$$\text{trace } \hat{S}_t = 1 - t(\text{trace } S) = 1,$$

because  $S$  has trace 0. Hence all these matrices satisfy condition (i) of Lemma 3.5.

Let us look for the possible values of  $t$  that give solutions of Lemma 3.5. That is, we want  $t > 0$  for which one eigenvalue  $\lambda_i(t)$  is 0 and the others are such that their sum is 1 and their product is  $\geq 0$  and  $\leq \frac{1}{4}$ .

Let us number the eigenvalues of  $S$  so that  $\mu_1 \leq \mu_2 \leq \mu_3$ . Since their sum is 0 and  $S$  is not the zero matrix, one must have  $\mu_1 < 0$  and  $\mu_3 > 0$ . If we want  $t$  as above, one  $\lambda_i(t)$  must vanish. Let us look for solutions with  $\lambda_1(t) = 0$ . This means that  $t = \frac{1}{3\mu_1} < 0$ , and we want  $t > 0$ . Hence, there are no solutions with  $\lambda_1(t) = 0$ .

Now let us look for solutions with  $\lambda_2(t) = 0$ . This implies that  $t = \frac{1}{3\mu_2}$ ; for this to be possible we must have  $\mu_2 \neq 0$ . If  $\mu_2 < 0$ , then  $t < 0$  and we want  $t$  to be positive. Thus, we only care about  $\mu_2 > 0$ . We have

$$\lambda_1(t) = \frac{1}{3}\left(1 - \frac{\mu_1}{\mu_2}\right) \quad \text{and} \quad \lambda_3(t) = \frac{1}{3}\left(1 - \frac{\mu_3}{\mu_2}\right).$$

We have  $\mu_1 < 0 < \mu_2$ , so  $\lambda_1(t) > 0$ . If  $\mu_2 < \mu_3$ , then  $\lambda_3(t) < 0$ , thus the product  $\lambda_1(t)\lambda_3(t)$  is  $< 0$ , so there are no such solutions to Lemma 3.8. The

other possibility is  $\mu_2 = \mu_3$ ; this also implies  $\lambda_3(t) = 0$ . In this case one has  $\lambda_1(t) = 1$  and  $\lambda_2(t) = \lambda_3(t) = 0$ , and  $t = \frac{1}{3\mu_2}$  is positive. Hence we have a solution, and this is unique because  $\mu_2 = \mu_3$ . If  $\mu_2 = 0$ , then  $\lambda_2(t)$  cannot be 0 and we cannot find solutions like this.

Summarizing, so far we have seen that: i) there are no solutions as in Lemma 3.8 for which  $\lambda_1(t) = 0$ ; ii) if  $\mu_2 \leq 0$ , there are no solutions as in Lemma 3.8 for which  $\lambda_2(t) = 0$ ; and iii) if  $\mu_2 = \mu_3$ , then there is a unique solution as in Lemma 3.8, for which  $\lambda_2(t) = \lambda_3(t) = 0$  and  $\lambda_1(t) = 1$ .

Finally, let us look for solutions with  $\lambda_3(t) = 0$ , i.e. with  $t = \frac{1}{3\mu_3}$ . We know, by hypothesis, that  $\mu_2 \leq \mu_3$  and  $\mu_3 > 0$ . If  $\mu_2 = \mu_3$ , then we are in the previous case and there is a unique positive  $t$  giving a solution as in Lemma 3.8. Let us assume now that  $\mu_2 < \mu_3$ . Then we have

$$\lambda_1(t) = \frac{1}{3}\left(1 - \frac{\mu_1}{\mu_3}\right) \quad \text{and} \quad \lambda_2(t) = \frac{1}{3}\left(1 - \frac{\mu_2}{\mu_3}\right),$$

which are both  $\geq 0$ . Since their sum is 1, it follows that each  $\lambda_i(t)$  is also  $\leq 1$ .

The product of  $\lambda_1(t)$  and  $\lambda_2(t)$  satisfies

$$\begin{aligned} 0 \leq \lambda_1(t) \cdot \lambda_2(t) &= \frac{1}{9}\left(1 - \frac{\mu_1 + \mu_2}{\mu_3} + \frac{\mu_1\mu_2}{\mu_3^2}\right) = \frac{1}{9}\left(2 + \frac{\mu_1\mu_2}{\mu_3^2}\right) \\ &= \frac{1}{9}\left(2 + \frac{\mu_1\mu_2}{(\mu_1 + \mu_2)^2}\right) \leq \frac{1}{4}, \end{aligned}$$

since  $\mu_1 + \mu_2 + \mu_3 = 0$  and  $\frac{\mu_1\mu_2}{(\mu_1 + \mu_2)^2} \leq \frac{1}{4}$  because  $\frac{1}{4}(a + b)^2 \geq ab$  for any real numbers  $a$  and  $b$  (with equality if and only if  $a = b$ ). Hence  $t = \frac{1}{3\mu_3}$  is the unique solution satisfying the conditions of Lemma 3.8.  $\square$

We now “normalize” the map  $\psi$  so that its image is contained in  $S^4 \subset \mathcal{S}$ . For this we define a function

$$\alpha(H) = [\text{trace}(\psi(H)^2)]^{-\frac{1}{2}},$$

i.e.  $\alpha(H)$  is the inverse of the norm of  $\psi(H)$  in  $\mathcal{S}$ , and we set

$$\Phi(H) = \alpha(H) \psi(H).$$

One has

$$\begin{aligned} \text{trace}[\psi(H)^2] &= \text{trace}\left[\left(\frac{1}{3}I_3 - \frac{1}{2}(H + \bar{H})\right)^2\right] \\ &= \text{trace}\left[\frac{1}{9}I_3 - \frac{1}{3}(H + \bar{H}) + \frac{1}{4}(H^2 + \bar{H}^2 + H\bar{H} + \bar{H}H)\right] \\ &= \frac{1}{6} + \frac{1}{4}\text{trace}(H\bar{H} + \bar{H}H), \end{aligned}$$

which is always positive since the matrix  $(H\bar{H} + \bar{H}H)$  is positive semi-definite, so its trace is  $\geq 0$ . Hence the maps  $\alpha$  and  $\Phi$  are well defined. It is clear that the image of  $\Phi$  is contained in  $S^4 \subset S$ , because the linearity of the trace implies that

$$[\text{trace}(\Phi(H))]^2 = \alpha^2(H) [\text{trace} \psi(H)]^2 = 1.$$

It is also clear that  $\Phi$  is  $\text{SO}(3, \mathbf{R})$ -equivariant, since the trace is invariant under conjugation and  $\psi$  is equivariant by Lemma 3.7. These considerations imply both Lemma 3.8 and the following

LEMMA 3.9. *The map  $\Phi$  is an equivariant surjection from  $P(2)$  over  $S^4 \subset S$ , and it is two-to-one, except over the image of the real matrices in  $P(2)$  where it is one-to-one.*

This gives the map in Theorem 3.4 that determines an equivariant diffeomorphism between  $S^4$  and  $P_{\mathbf{C}}^2$  modulo the involution given by conjugation. To complete the proof of Theorem 3.4 we need to show that  $\Phi$  is invariant under the involution of  $P(2)$  that corresponds to complex conjugation in  $P_{\mathbf{C}}^2$ . For this we notice that if  $L_H$  is the complex line in  $\mathbf{C}^3$  which is the image of  $H \in P(2)$ , and if  $0 \neq (z_1, z_2, z_3) \in L_H$ , we can associate to  $H$  the point in  $P_{\mathbf{C}}^2$  with projective coordinates  $[z_1, z_2, z_3]$ . To the matrix  $\bar{H}$  there corresponds the line with projective coordinates  $[\bar{z}_1, \bar{z}_2, \bar{z}_3]$ . Therefore we have

LEMMA 3.10. *The involution  $j^*$  of  $P(2)$  defined by  $j^*(H) = \bar{H}$  coincides with the involution  $j$  of  $P_{\mathbf{C}}^2$  given by complex conjugation,  $[z_1, z_2, z_3] \xrightarrow{j} [\bar{z}_1, \bar{z}_2, \bar{z}_3]$ .*

Then  $\Phi$  is invariant under this involution, since  $\Re(H) = \Re(\bar{H})$ , proving Theorem 3.4.  $\square$

#### 4. SOME APPLICATIONS AND REMARKS

It is interesting to describe explicitly the orbits of the  $\Gamma$  action of  $\text{SO}(3, \mathbf{R})$  on  $S^4$ , regarded<sup>2)</sup> as the set of matrices with norm 1 in  $S$ . In fact, the orbits of this action are conjugacy classes (or *congruency classes*) of traceless symmetric matrices whose square has trace 1. This is the connection between our construction and the spherical Tits buildings. Every  $S \in S$  can

<sup>2)</sup> This orbit description of  $S^4$  is also given in [Ma2].