Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

**Band:** 49 (2003)

**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: QUADRICS, ORTHOGONAL ACTIONS AND INVOLUTIONS IN

COMPLEX PROJECTIVE SPACES

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Kapitel: 3. \$P C^2\$ AND THE 4-SPHERE \$S^4\$

**DOI:** https://doi.org/10.5169/seals-66686

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that we considered in Sections 2 and 3 above. Similarly, if G is  $\mathbb{Z}/2\mathbb{Z}$  acting on Q as the antipodal map, then the corresponding extension to  $P_{\mathbb{C}}^2$  is given by complex conjugation.

# 3. $P_{\mathbf{C}}^2$ and the 4-sphere $S^4$

The previous discussion, restricted to n=2 and compared to the cohomogeneity 1 isometric action of  $SO(3,\mathbf{R})$  on  $S^4$  constructed in [HL], motivates an equivariant version of the Arnold-Kuiper-Massey theorem [Ar1, Ar2, Ku, Ma1], saying that  $P_{\mathbf{C}}^2$  modulo conjugation is the 4-sphere. In this section we give a new proof of this theorem. We construct an explicit algebraic map  $\Phi\colon P_{\mathbf{C}}^2\to S^4$ , which is equivariant with respect to the cohomogeneity 1 isometric actions of  $SO(3,\mathbf{R})$  on  $P_{\mathbf{C}}^2$  and  $S^4$  and induces a diffeomorphism  $P_{\mathbf{C}}^2/conjugation\cong S^4$ .

We start by recalling the  $SO(3, \mathbf{R})$ -action on  $S^4$ , as explained by Hsiang and Lawson in [HL; Example 1.4].

Let S be the vector space of real  $3 \times 3$ , traceless and symmetric matrices. As a real vector space S is  $\mathbb{R}^5$ , and it can be equipped with a metric given by the inner product  $(A,B) \mapsto \operatorname{trace}(AB)$ . Let  $S^{(4)}$  be the space of matrices in S with norm 1. One has an obvious diffeomorphism  $S^4 \cong S^{(4)}$ , which becomes isometric if we endow  $S^4$  with its usual round metric and  $S^{(4)}$  with the metric given by the inner product in S. We shall identify these two spaces in the sequel, denoting both of them by  $S^4$  indistinctly. The group  $SO(3,\mathbb{R})$  acts on S by  $A \mapsto O^t AO$ , where  $O^t$  is the transposed matrix (which is equal, in our case, to  $O^{-1}$ ). This induces an isometric action  $\Gamma$  of  $SO(3,\mathbb{R})$  on  $S^4$ . This action on  $S^4$  has two disjoint copies of  $P^2_{\mathbb{R}}$  as special fibres (see the remark at the end of this section). The space of orbits is the interval [0,1], with the endpoints giving the special orbits. Each principal orbit (i.e. the orbits of highest dimension) is a flag manifold

$$F^3(2,1) \cong \operatorname{SO}(3,\mathbf{R}) / (\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}) \cong L(4,1) / (\mathbf{Z}/2\mathbf{Z}),$$

of pairs (P, l) with P a plane in  $\mathbb{R}^3$  and l line in P, where L(4, 1) is the lens space  $S^3/(\mathbb{Z}/4\mathbb{Z})\cong SO(3, \mathbb{R})/(\mathbb{Z}/2\mathbb{Z})$ .

Let us give a similar description of  $P_{\mathbb{C}}^2$ . Let

$$\mathfrak{H}(3,\mathbf{C}) = \{ H \in M(3,\mathbf{C}) \mid H = H^* \}$$

be the space of complex  $3 \times 3$  Hermitian matrices, where  $H^* = \overline{H}^t$  is the adjoint matrix of H, obtained by first conjugating each entry of H and then

transposing the matrix. We equip  $\mathfrak{H}(3, \mathbb{C})$  with the Hermitian inner product

(3.1) 
$$\langle H_1, H_2 \rangle = \frac{1}{2} \operatorname{trace} (H_1 H_2).$$

As a vector space, with this inner product,  $\mathfrak{H}(3, \mathbb{C})$  is the ordinary Euclidean space  $\mathbb{E}^9$ . Consider the subset P(2) of  $\mathfrak{H}(3, \mathbb{C})$  defined by

(3.2) 
$$P(2) = \{ H \in \mathfrak{H}(3, \mathbb{C}) \mid H^2 = H \text{ and } trace(H) = 1 \}.$$

LEMMA 3.3. The set P(2) is a manifold, diffeomorphic to  $P_{\mathbb{C}}^2$ . Moreover, if we endow P(2) with the metric defined by (3.1), then P(2) is isometric to  $P_{\mathbb{C}}^2$  equipped with the Fubini-Study metric (of constant holomorphic sectional curvature 4).

We remark that it is possible to describe  $P_{\mathbb{C}}^n$  in a similar way, but we restrict our attention to n=2 because this is all we need.

*Proof.* We claim that if H is in P(2), then it is an orthogonal projection over a complex line. In fact, if H is in P(2), then it is diagonalizable by a unitary matrix and its eigenvalues are 0 or 1, because  $H^2 = H$ . Since the trace is one, two eigenvalues must be 0 and the other is 1. Hence H is a surjection of  $\mathbb{C}^3$  over a complex line, and this map has to be an orthogonal projection because H is Hermitian. Conversely, it is clear that each line  $L \in \mathbb{C}^3$  determines a unique orthogonal projection of  $\mathbb{C}^3$ , and this is given by a matrix in P(2). The diffeomorphism in Lemma 3.3 is achieved by the map that carries H into the corresponding line in  $\mathbb{C}^3$ . To prove that this map gives a metric equivalence, we notice that the unitary group U(3) acts on  $\mathfrak{H}(3,\mathbb{C})$  by  $H \mapsto U^*HU$ , and P(2) is an orbit of this action, with isotropy  $U(2) \times U(1)$ . Thus,

$$P(2) \cong U(3)/(U(2) \times U(1)) \cong P_{\mathbf{C}}^2$$

and the metric on P(2) is obviously U(3)-invariant. Hence the induced metric on  $P_{\mathbf{C}}^2$  is also U(3)-invariant, and this characterizes the Fubini-Study metric, up to scaling.

We recall now that the quotient of  $P_{\mathbb{C}}^2$  by the complex conjugation j is a smooth manifold, which is not an obvious fact since j has fixed points. This is carefully explained in [Mar], so we only sketch a few ideas here. Away from the fixed point set  $\Pi \cong P_{\mathbb{R}}^2$ , the involution j is free, so the quotient is a smooth manifold. The problem is on  $\Pi$ . A tubular neighbourhood of

 $\Pi$  in  $P_{\mathbf{C}}^2$  can be regarded as an open disk normal bundle, and conjugation carries each normal fibre into itself. Since the quotient of each normal 2-disk by the involution is again a 2-disk, it follows that the quotient  $P_{\mathbf{C}}^2/j$  is a topological manifold. Making this argument more carefully one gets that  $P_{\mathbf{C}}^2/j$  is in fact a PL-manifold, as noticed in [Ku], and therefore it is smooth, since every piecewise linear 4-manifold is smooth. In [Mar] Marin defines the smooth structure on  $P_{\mathbf{C}}^2/j$  directly, without using PL-structures. An important point is that the smooth structure on  $P_{\mathbf{C}}^2/j$  is such that the obvious projection  $P_{\mathbf{C}}^2 \to P_{\mathbf{C}}^2/j$  is differentiable.

Let us denote by  $\Gamma$  the aforementioned isometric action of  $SO(3, \mathbf{R})$  on  $S^4$ , and by  $\widetilde{\Gamma}$  the standard action of  $SO(3, \mathbf{R})$  on  $P_{\mathbf{C}}^2$ , which is by isometries with respect to the Fubini-Study metric. This action is defined either by considering  $SO(3, \mathbf{R})$  as a subgroup of  $O(3, \mathbf{C})$ , acting on the space of lines in  $\mathbf{C}^3$ , or via the action of  $SO(3, \mathbf{R})$  on the space of matrices  $P(2) \subset H(3, \mathbf{C})$  given by

$$(O,A)\mapsto O^tAO$$
.

By Lemma 3.3, both metrics on  $P_{\mathbf{C}}^2$  are equivalent; also for every  $O \in \mathrm{SO}(3,\mathbf{R}), \ H \in P(2)$  and  $v \in \mathbf{C}^3$  such that H(v) = v, one has  $O^t HO(O^{-1}(v)) = O^{-1}(v)$ , because  $O^{-1} = O^t$ . Hence both actions on  $P_{\mathbf{C}}^2 \cong P(2)$  are equivalent. Similarly, given the  $\mathrm{SO}(3,\mathbf{R})$ -actions  $\widetilde{\Gamma}$  on  $P_{\mathbf{C}}^2$  and  $\Gamma$  on  $S^4$ , we say that these actions are equivariant if there exists a map  $\Phi \colon P_{\mathbf{C}}^2 \to S^4$  which makes the following diagram commutative:

$$SO(3, \mathbf{R}) \times P_{\mathbf{C}}^{2} \xrightarrow{\widetilde{\Gamma}} P_{\mathbf{C}}^{2}$$

$$Id \times \Phi \downarrow \qquad \qquad \Phi \downarrow$$

$$SO(3, \mathbf{R}) \times S^{4} \xrightarrow{\Gamma} S^{4}.$$

In this case we say that  $\Phi$  conjugates the actions  $\Gamma$  and  $\widetilde{\Gamma}$ . The map  $\Phi$  carries orbits into orbits, i.e. the decompositions of  $P_{\mathbb{C}}^2$  and  $S^4$  into orbits are (smoothly) equivalent.

Let us now state the equivariant Arnold-Kuiper-Massey theorem:

THEOREM 3.4. There is a real algebraic equivariant map  $\Phi: P_{\mathbf{C}}^2 \to S^4$ , which is invariant by the complex conjugation j and induces a diffeomorphism  $P_{\mathbf{C}}^2/j \cong S^4$ , providing a conjugation between the isometric  $\mathrm{SO}(3,\mathbf{R})$ -actions  $\widetilde{\Gamma}$  on  $P_{\mathbf{C}}^2$  and  $\Gamma$  on  $S^4$ .

We notice that Theorem 3.4, together with [HL], imply that the image of  $P_{\mathbf{R}}^2 \subset P_{\mathbf{C}}^2$  under the above map is the image of  $P_{\mathbf{R}}^2$  by the classical Veronese embedding  $(P_{\mathbf{C}}^2, P_{\mathbf{R}}^2) \hookrightarrow (P_{\mathbf{C}}^5, S^4)$ .

The proof of Theorem 3.4 follows from several lemmas below.

LEMMA 3.5. Let A be a real  $(3 \times 3)$ -matrix. Then A is the real part of a matrix H in P(2) if and only if

- i) A is symmetric with trace 1;
- ii) A has 0 as an eigenvalue and the other two eigenvalues  $\lambda_i$  and  $\lambda_j$  are roots of an equation of the form:

$$\lambda^2 - \lambda + k = 0$$

for some constant  $k \in \mathbf{R}$  with  $0 \le k \le \frac{1}{4}$ .

If A and H are as above, and if  $O \in SO(3, \mathbb{R})$  is such that  $O^tAO$  is a diagonal matrix, then the imaginary part B of H, taken into its canonical form  $O^tBO$ , has only two possible non-zero entries, which are  $\pm \sqrt{k}$ . In particular, if k = 0, then H = A.

*Proof.* Let us consider a matrix  $H \in P(2)$  and decompose it into its real and imaginary parts: H = A + iB. Then one has  $\overline{H}^t = A^t - iB^t$ . Also  $H = \overline{H}^t$  because H is Hermitian. Hence  $A = A^t$  and  $B = -B^t$ , i.e. A is symmetric and B is anti-symmetric. Thus the trace of A is 1, proving statement (i). One also has

$$H^2 = A^2 - B^2 + i(AB + BA),$$

and  $H^2 = H$  because H is in P(2). Therefore  $A = A^2 - B^2$  and B = AB + BA.

Now, A is symmetric, and so is  $A^2$ ; these two matrices obviously commute, so they can be diagonalized simultaneously by a matrix  $O \in SO(3, \mathbf{R})$ . Since  $B^2 = A^2 - A$ , one knows that  $O^t B^2 O$  is also diagonal:

$$O^t B^2 O = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix} ,$$

with  $\mu_i = \lambda_i^2 - \lambda_i$ , for each i = 1, 2, 3, where the  $\lambda_i$  are the eigenvalues of A. But B is antisymmetric and commutes with  $B^2$ , which is symmetric. Hence the same matrix O takes B to its canonical form:

$$O^t B O = \begin{pmatrix} 0 & a & c \\ -a & 0 & b \\ -c & -b & 0 \end{pmatrix}$$

for some  $a, b, c \in \mathbb{C}$ . This implies that

$$O^{t}B^{2}O = (O^{t}BO)(O^{t}BO) = \begin{pmatrix} -a^{2} - c^{2} & -bc & ab \\ -bc & -a^{2} - b^{2} & -ac \\ ab & -ac & -b^{2} - c^{2} \end{pmatrix},$$

which we know is a diagonal matrix. Therefore two of the numbers a, b, c must be zero. Assume for instance that a and b are 0, then both eigenvalues  $\lambda_1$  and  $\lambda_3$  are roots of the polynomial

$$\lambda^2 - \lambda + c^2 = 0.$$

This implies that

$$\lambda_1 + \lambda_3 = 1$$
 and  $\lambda_1 \cdot \lambda_3 = c^2 \ge 0$ .

Hence  $\lambda_2 = 0$  (because the trace of A is 1), so 0 is an eigenvalue of A. The other eigenvalues  $\lambda_1$  and  $\lambda_3$  must both be  $\geq 0$  and  $\leq 1$ , because their product is non-negative and their sum is 1. Moreover the roots must be real, therefore  $k = c^2 \leq \frac{1}{4}$ , proving statement (ii).

Also, in this case the eigenvalues of A determine the imaginary part B of H up to sign:

$$B = \pm O \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ -c & 0 & 0 \end{pmatrix} O^{t},$$

with  $c^2 = \lambda_1 - \lambda_3^2 = \lambda_3 - \lambda_3^2$ , proving in this case the last statement of Lemma 3.5. The other cases, when either a = c = 0 or b = c = 0, are similar to the previous one. This proves that if  $A = \Re(H)$  for some matrix  $H \in P(2)$ , then A is as stated in Lemma 3.5. Conversely, given A satisfying these conditions, the above arguments tell us how to construct B so that these matrices are the real and imaginary parts of some H in P(2).  $\square$ 

Now, given  $H \in P(2)$ , its real part is  $\Re(H) = \frac{1}{2}(H + \overline{H})$ . Define

$$\psi \colon P(2) \to M(3, \mathbf{R})$$
,

the space  $M(3, \mathbf{R})$  being the space of real  $(3 \times 3)$ -matrices, by the formula

(3.6) 
$$\psi(H) = \frac{1}{3} I_3 - \Re(H) \in M(3, \mathbf{R}),$$

where  $I_3$  is the  $(3 \times 3)$ -identity matrix. In other words,  $\psi(H)$  is the real part of the matrix  $(\frac{1}{3}I_3 - H)$ . Since  $H \in P(2)$ , it follows that  $\psi(H)$  is actually contained in S.

It is clear that the above action of  $SO(3, \mathbf{R})$  on P(2) given by conjugation is equivalent, via the above diffeomorphism  $P(2) \cong P_{\mathbf{C}}^2$ , with the standard action

studied in §2 and §3 above. It is also clear that, for every  $O \in SO(3, \mathbb{R})$ , one has

$$\psi(O^t H O) = \frac{1}{3} I - \frac{1}{2} \left( O^t (H + \overline{H}) O \right) = O^t \left( \frac{1}{3} I - \frac{1}{2} (H + \overline{H}) \right) O = O^t \psi(H) O.$$

Hence we have

LEMMA 3.7. The map  $\psi$  is equivariant. That is, for every  $O \in SO(3, \mathbf{R})$  and  $H \in P(2)$ , one has  $\psi(O^tHO) = O^t\psi(H)O$ .

LEMMA 3.8. Given  $S \in S - \{0\}$ , there exists a unique positive  $t \in \mathbb{R}$ , such that the matrix  $(\frac{1}{3}I - tS)$  is the real part of some matrix  $H \in P(2)$ .

*Proof.* By Lemma 3.7, we may assume that S is diagonal. Hence the matrix  $\widehat{S}_t = (\frac{1}{3}I - tS)$  is also diagonal, say

$$\widehat{S}_t = \begin{pmatrix} \lambda_1(t) & 0 & 0 \\ 0 & \lambda_2(t) & 0 \\ 0 & 0 & \lambda_3(t) \end{pmatrix}$$

with  $\lambda_i(t) = \frac{1}{3} - t\mu_i$ , where the  $\mu_i$  are the eigenvalues of S. We notice that for all  $t \in \mathbf{R}$ , one has

trace 
$$\widehat{S}_t = 1 - t \text{ (trace } S) = 1$$
,

because S has trace 0. Hence all these matrices satisfy condition (i) of Lemma 3.5.

Let us look for the possible values of t that give solutions of Lemma 3.5. That is, we want t > 0 for which one eigenvalue  $\lambda_i(t)$  is 0 and the others are such that their sum is 1 and their product is  $\geq 0$  and  $\leq \frac{1}{4}$ .

Let us number the eigenvalues of S so that  $\mu_1 \leq \mu_2 \leq \mu_3$ . Since their sum is 0 and S is not the zero matrix, one must have  $\mu_1 < 0$  and  $\mu_3 > 0$ . If we want t as above, one  $\lambda_i(t)$  must vanish. Let us look for solutions with  $\lambda_1(t) = 0$ . This means that  $t = \frac{1}{3\mu_1} < 0$ , and we want t > 0. Hence, there are no solutions with  $\lambda_1(t) = 0$ .

Now let us look for solutions with  $\lambda_2(t)=0$ . This implies that  $t=\frac{1}{3\mu_2}$ ; for this to be possible we must have  $\mu_2\neq 0$ . If  $\mu_2<0$ , then t<0 and we want t to be positive. Thus, we only care about  $\mu_2>0$ . We have

$$\lambda_1(t) = \frac{1}{3}(1 - \frac{\mu_1}{\mu_2})$$
 and  $\lambda_3(t) = \frac{1}{3}(1 - \frac{\mu_3}{\mu_2})$ .

We have  $\mu_1 < 0 < \mu_2$ , so  $\lambda_1(t) > 0$ . If  $\mu_2 < \mu_3$ , then  $\lambda_3(t) < 0$ , thus the product  $\lambda_1(t)\lambda_3(t)$  is < 0, so there are no such solutions to Lemma 3.8. The

other possibility is  $\mu_2 = \mu_3$ ; this also implies  $\lambda_3(t) = 0$ . In this case one has  $\lambda_1(t) = 1$  and  $\lambda_2(t) = \lambda_3(t) = 0$ , and  $t = \frac{1}{3\mu_2}$  is positive. Hence we have a solution, and this is unique because  $\mu_2 = \mu_3$ . If  $\mu_2 = 0$ , then  $\lambda_2(t)$  cannot be 0 and we cannot find solutions like this.

Summarizing, so far we have seen that: i) there are no solutions as in Lemma 3.8 for which  $\lambda_1(t)=0$ ; ii) if  $\mu_2\leq 0$ , there are no solutions as in Lemma 3.8 for which  $\lambda_2(t)=0$ ; and iii) if  $\mu_2=\mu_3$ , then there is a unique solution as in Lemma 3.8, for which  $\lambda_2(t)=\lambda_3(t)=0$  and  $\lambda_1(t)=1$ .

Finally, let us look for solutions with  $\lambda_3(t)=0$ , i.e. with  $t=\frac{1}{3\mu_3}$ . We know, by hypothesis, that  $\mu_2 \leq \mu_3$  and  $\mu_3 > 0$ . If  $\mu_2 = \mu_3$ , then we are in the previous case and there is a unique positive t giving a solution as in Lemma 3.8. Let us assume now that  $\mu_2 < \mu_3$ . Then we have

$$\lambda_1(t) = \frac{1}{3}(1 - \frac{\mu_1}{\mu_3})$$
 and  $\lambda_2(t) = \frac{1}{3}(1 - \frac{\mu_2}{\mu_3})$ ,

which are both  $\geq 0$ . Since their sum is 1, it follows that each  $\lambda_i(t)$  is also  $\leq 1$ .

The product of  $\lambda_1(t)$  and  $\lambda_2(t)$  satisfies

$$0 \le \lambda_1(t) \cdot \lambda_2(t) = \frac{1}{9} \left(1 - \frac{\mu_1 + \mu_2}{\mu_3} + \frac{\mu_1 \mu_2}{\mu_3^2}\right) = \frac{1}{9} \left(2 + \frac{\mu_1 \mu_2}{\mu_3^2}\right)$$
$$= \frac{1}{9} \left(2 + \frac{\mu_1 \mu_2}{(\mu_1 + \mu_2)^2}\right) \le \frac{1}{4},$$

since  $\mu_1 + \mu_2 + \mu_3 = 0$  and  $\frac{\mu_1 \mu_2}{(\mu_1 + \mu_2)^2} \le \frac{1}{4}$  because  $\frac{1}{4}(a+b)^2 \ge ab$  for any real numbers a and b (with equality if and only if a = b). Hence  $t = \frac{1}{3\mu_3}$  is the unique solution satisfying the conditions of Lemma 3.8.  $\square$ 

We now "normalize" the map  $\psi$  so that its image is contained in  $S^4\subset \mathcal{S}$ . For this we define a function

$$\alpha(H) = [\operatorname{trace}(\psi(H)^2)]^{-\frac{1}{2}},$$

i.e.  $\alpha(H)$  is the inverse of the norm of  $\psi(H)$  in S, and we set

$$\Phi(H) = \alpha(H) \, \psi(H) \, .$$

One has

$$trace[\psi(H)^{2}] = trace[(\frac{1}{3}I_{3} - \frac{1}{2}(H + \overline{H}))^{2}]$$

$$= trace[\frac{1}{9}I_{3} - \frac{1}{3}(H + \overline{H}) + \frac{1}{4}(H^{2} + \overline{H}^{2} + H\overline{H} + \overline{H}H)]$$

$$= \frac{1}{6} + \frac{1}{4}trace(H\overline{H} + \overline{H}H),$$

which is always positive since the matrix  $(H\overline{H}+\overline{H}H)$  is positive semi-definite, so its trace is  $\geq 0$ . Hence the maps  $\alpha$  and  $\Phi$  are well defined. It is clear that the image of  $\Phi$  is contained in  $S^4 \subset \mathcal{S}$ , because the linearity of the trace implies that

 $[\operatorname{trace}(\Phi(H))]^2 = \alpha^2(H) [\operatorname{trace} \psi(H)]^2 = 1$ .

It is also clear that  $\Phi$  is SO(3, **R**)-equivariant, since the trace is invariant under conjugation and  $\psi$  is equivariant by Lemma 3.7. These considerations imply both Lemma 3.8 and the following

LEMMA 3.9. The map  $\Phi$  is an equivariant surjection from P(2) over  $S^4 \subset S$ , and it is two-to-one, except over the image of the real matrices in P(2) where it is one-to-one.

This gives the map in Theorem 3.4 that determines an equivariant diffeomorphism between  $S^4$  and  $P_{\mathbb{C}}^2$  modulo the involution given by conjugation. To complete the proof of Theorem 3.4 we need to show that  $\Phi$  is invariant under the involution of P(2) that corresponds to complex conjugation in  $P_{\mathbb{C}}^2$ . For this we notice that if  $L_H$  is the complex line in  $\mathbb{C}^3$  which is the image of  $H \in P(2)$ , and if  $0 \neq (z_1, z_2, z_3) \in L_H$ , we can associate to H the point in  $P_{\mathbb{C}}^2$  with projective coordinates  $[z_1, z_2, z_3]$ . To the matrix  $\overline{H}$  there corresponds the line with projective coordinates  $[\overline{z}_1, \overline{z}_2, \overline{z}_3]$ . Therefore we have

LEMMA 3.10. The involution j\* of P(2) defined by  $j*(H) = \overline{H}$  coincides with the involution j of  $P_{\mathbf{C}}^2$  given by complex conjugation,  $[z_1, z_2, z_3] \stackrel{j}{\mapsto} [\overline{z}_1, \overline{z}_2, \overline{z}_3]$ .

Then  $\Phi$  is invariant under this involution, since  $\Re(H) = \Re(\overline{H})$ , proving Theorem 3.4.  $\square$ 

## 4. Some applications and remarks

It is interesting to describe explicitly the orbits of the  $\Gamma$  action of  $SO(3, \mathbf{R})$  on  $S^4$ , regarded<sup>2</sup>) as the set of matrices with norm 1 in S. In fact, the orbits of this action are conjugacy classes (or *congruency classes*) of traceless symmetric matrices whose square has trace 1. This is the connection between our construction and the spherical Tits buildings. Every  $S \in S$  can

<sup>&</sup>lt;sup>2</sup>) This orbit description of  $S^4$  is also given in [Ma2].