Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 49 (2003)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: QUADRICS, ORTHOGONAL ACTIONS AND INVOLUTIONS IN COMPLEX PROJECTIVE SPACES

Autor: Tráng, LÊ Dng / Seade, José / Verjovsky, Alberto

Kapitel: 3. P_C^2 AND THE 4-SPHERE S^4

DOI: https://doi.org/10.5169/seals-66686

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 15.02.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

that we considered in Sections 2 and 3 above. Similarly, if G is $\mathbf{Z}/2\mathbf{Z}$ acting on Q as the antipodal map, then the corresponding extension to $P^2_{\mathbf{C}}$ is given by complex conjugation.

3. $P^2_{\mathbf{C}}$ AND THE 4-SPHERE S^4

The previous discussion, restricted to $n = 2$ and compared to the cohomogeneity 1 isometric action of $\mathrm{SO}(3,\mathbf{R})$ on S^4 constructed in [HL], motivates an equivariant version of the Arnold-Kuiper-Massey theorem [Ar1, Ar2, Ku, Ma1], saying that $P^2_{\mathbf{C}}$ modulo conjugation is the 4-sphere. In this section we give a new proof of this theorem. We construct an explicit algebraic map $\Phi\colon P^2_{\mathbf{C}} \to S^4$, which is equivariant with respect to the cohomogeneity 1 isometric actions of $\mathrm{SO}(3,\mathbf{R})$ on $P^2_{\mathbf{C}}$ and S^4 and induces a diffeomorphism $P^2_{\mathbf{C}}/\mathit{conjugation} \cong S^4$.

We start by recalling the $\mathrm{SO}(3,\mathbf{R})$-action on S^4, as explained by Hsiang and Lawson in [HL; Example 1.4].

Let $\mathcal{S}$ be the vector space of real 3×3, traceless and symmetric matrices. As a real vector space $\mathcal{S}$ is $\mathbf{R}^5$, and it can be equipped with a metric given by the inner product $(A,B) \mapsto \mathrm{trace}(AB)$. Let $\mathcal{S}^{(4)}$ be the space of matrices in $\mathcal{S}$ with norm 1. One has an obvious diffeomorphism $S^4 \cong \mathcal{S}^{(4)}$, which becomes isometric if we endow S^4 with its usual round metric and $\mathcal{S}^{(4)}$ with the metric given by the inner product in $\mathcal{S}$. We shall identify these two spaces in the sequel, denoting both of them by S^4 indistinctly. The group $\mathrm{SO}(3,\mathbf{R})$ acts on $\mathcal{S}$ by $A \mapsto O^tAO$, where O^t is the transposed matrix (which is equal, in our case, to O^{-1}). This induces an isometric action Γ of $\mathrm{SO}(3,\mathbf{R})$ on S^4. This action on S^4 has two disjoint copies of $P^2_{\mathbf{R}}$ as special fibres (see the remark at the end of this section). The space of orbits is the interval $[0,1]$, with the endpoints giving the special orbits. Each principal orbit (i.e. the orbits of highest dimension) is a flag manifold

$$F^3(2,1) \cong \mathrm{SO}(3,\mathbf{R}) \,/\, (\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}) \cong L(4,1) \,/\, (\mathbf{Z}/2\mathbf{Z}),$$

of pairs (P,l) with P a plane in $\mathbf{R}^3$ and l line in P, where $L(4,1)$ is the lens space $S^3 \,/\, (\mathbf{Z}/4\mathbf{Z}) \cong \mathrm{SO}(3,\mathbf{R}) \,/\, (\mathbf{Z}/2\mathbf{Z})$.

Let us give a similar description of $P^2_{\mathbf{C}}$. Let

$$\mathfrak{H}(3,\mathbf{C}) = \{H \in M(3,\mathbf{C}) \mid H = H^*\}$$

be the space of complex 3×3 Hermitian matrices, where $H^* = \overline{H}^t$ is the adjoint matrix of H, obtained by first conjugating each entry of H and then

transposing the matrix. We equip $\mathfrak{H}(3, \mathbf{C})$ with the Hermitian inner product

$$\langle H_1, H_2 \rangle = \frac{1}{2} \operatorname{trace}(H_1 H_2). \tag{3.1}$$

As a vector space, with this inner product, $\mathfrak{H}(3, \mathbf{C})$ is the ordinary Euclidean space $\mathbf{E}^9$. Consider the subset $P(2)$ of $\mathfrak{H}(3, \mathbf{C})$ defined by

$$P(2) = \left\{ H \in \mathfrak{H}(3, \mathbf{C}) \mid H^2 = H \quad \text{and} \quad \operatorname{trace}(H) = 1 \right\}. \tag{3.2}$$

LEMMA 3.3. *The set $P(2)$ is a manifold, diffeomorphic to $P^2_{\mathbf{C}}$. Moreover, if we endow $P(2)$ with the metric defined by (3.1), then $P(2)$ is isometric to $P^2_{\mathbf{C}}$ equipped with the Fubini-Study metric (of constant holomorphic sectional curvature 4).*

We remark that it is possible to describe $P^n_{\mathbf{C}}$ in a similar way, but we restrict our attention to $n = 2$ because this is all we need.

Proof. We claim that if H is in $P(2)$, then it is an orthogonal projection over a complex line. In fact, if H is in $P(2)$, then it is diagonalizable by a unitary matrix and its eigenvalues are 0 or 1, because $H^2 = H$. Since the trace is one, two eigenvalues must be 0 and the other is 1. Hence H is a surjection of $\mathbf{C}^3$ over a complex line, and this map has to be an orthogonal projection because H is Hermitian. Conversely, it is clear that each line $L \in \mathbf{C}^3$ determines a unique orthogonal projection of $\mathbf{C}^3$, and this is given by a matrix in $P(2)$. The diffeomorphism in Lemma 3.3 is achieved by the map that carries H into the corresponding line in $\mathbf{C}^3$. To prove that this map gives a metric equivalence, we notice that the unitary group $U(3)$ acts on $\mathfrak{H}(3, \mathbf{C})$ by $H \mapsto U^* H U$, and $P(2)$ is an orbit of this action, with isotropy $\big(U(2) \times U(1)\big)$. Thus,

$$P(2) \cong U(3)/\big(U(2) \times U(1)\big) \cong P^2_{\mathbf{C}},$$

and the metric on $P(2)$ is obviously $U(3)$-invariant. Hence the induced metric on $P^2_{\mathbf{C}}$ is also $U(3)$-invariant, and this characterizes the Fubini-Study metric, up to scaling. $\square$

We recall now that the quotient of $P^2_{\mathbf{C}}$ by the complex conjugation j is a smooth manifold, which is not an obvious fact since j has fixed points. This is carefully explained in [Mar], so we only sketch a few ideas here. Away from the fixed point set $\Pi \cong P^2_{\mathbf{R}}$, the involution j is free, so the quotient is a smooth manifold. The problem is on Π. A tubular neighbourhood of

Π in $P^2_{\mathbf{C}}$ can be regarded as an open disk normal bundle, and conjugation carries each normal fibre into itself. Since the quotient of each normal 2-disk by the involution is again a 2-disk, it follows that the quotient $P^2_{\mathbf{C}}/j$ is a topological manifold. Making this argument more carefully one gets that $P^2_{\mathbf{C}}/j$ is in fact a PL-manifold, as noticed in [Ku], and therefore it is smooth, since every piecewise linear 4-manifold is smooth. In [Mar] Marin defines the smooth structure on $P^2_{\mathbf{C}}/j$ directly, without using PL-structures. An important point is that the smooth structure on $P^2_{\mathbf{C}}/j$ is such that the obvious projection $P^2_{\mathbf{C}} \to P^2_{\mathbf{C}}/j$ is differentiable.

Let us denote by Γ the aforementioned isometric action of $\mathrm{SO}(3,\mathbf{R})$ on S^4, and by $\widetilde{\Gamma}$ the standard action of $\mathrm{SO}(3,\mathbf{R})$ on $P^2_{\mathbf{C}}$, which is by isometries with respect to the Fubini-Study metric. This action is defined either by considering $\mathrm{SO}(3,\mathbf{R})$ as a subgroup of $O(3,\mathbf{C})$, acting on the space of lines in $\mathbf{C}^3$, or via the action of $\mathrm{SO}(3,\mathbf{R})$ on the space of matrices $P(2) \subset H(3,\mathbf{C})$ given by

$$(O,A) \mapsto O^t A O\,.$$

By Lemma 3.3, both metrics on $P^2_{\mathbf{C}}$ are equivalent; also for every $O \in \mathrm{SO}(3,\mathbf{R})$, $H \in P(2)$ and $v \in \mathbf{C}^3$ such that $H(v) = v$, one has $O^t H O(O^{-1}(v)) = O^{-1}(v)$, because $O^{-1} = O^t$. Hence both actions on $P^2_{\mathbf{C}} \cong P(2)$ are equivalent. Similarly, given the $\mathrm{SO}(3,\mathbf{R})$-actions $\widetilde{\Gamma}$ on $P^2_{\mathbf{C}}$ and Γ on S^4, we say that these actions are equivariant if there exists a map $\Phi\colon P^2_{\mathbf{C}} \to S^4$ which makes the following diagram commutative:

$$\begin{array}{ccc} \mathrm{SO}(3,\mathbf{R}) \times P^2_{\mathbf{C}} & \xrightarrow{\widetilde{\Gamma}} & P^2_{\mathbf{C}} \\ {\scriptstyle Id\times\Phi}\downarrow & & \downarrow{\scriptstyle \Phi} \\ \mathrm{SO}(3,\mathbf{R}) \times S^4 & \xrightarrow{\Gamma} & S^4\,. \end{array}$$

In this case we say that Φ *conjugates* the actions Γ and $\widetilde{\Gamma}$. The map Φ carries orbits into orbits, i.e. the decompositions of $P^2_{\mathbf{C}}$ and S^4 into orbits are (smoothly) equivalent.

Let us now state the equivariant Arnold-Kuiper-Massey theorem:

THEOREM 3.4. *There is a real algebraic equivariant map $\Phi\colon P^2_{\mathbf{C}} \to S^4$, which is invariant by the complex conjugation j and induces a diffeomorphism $P^2_{\mathbf{C}}/j \cong S^4$, providing a conjugation between the isometric $\mathrm{SO}(3,\mathbf{R})$-actions $\widetilde{\Gamma}$ on $P^2_{\mathbf{C}}$ and Γ on S^4.*

We notice that Theorem 3.4, together with [HL], imply that the image of $P^2_{\mathbf{R}} \subset P^2_{\mathbf{C}}$ under the above map is the image of $P^2_{\mathbf{R}}$ by the classical Veronese embedding $(P^2_{\mathbf{C}}, P^2_{\mathbf{R}}) \hookrightarrow (P^5_{\mathbf{C}}, S^4)$.

The proof of Theorem 3.4 follows from several lemmas below.

LEMMA 3.5. *Let A be a real (3×3)-matrix. Then A is the real part of a matrix H in $P(2)$ if and only if*

i) *A is symmetric with trace 1;*

ii) *A has 0 as an eigenvalue and the other two eigenvalues λ_i and λ_j are roots of an equation of the form:*

$$\lambda^2 - \lambda + k = 0,$$

for some constant $k \in \mathbf{R}$ with $0 \leq k \leq \frac{1}{4}$.

If A and H are as above, and if $O \in \mathrm{SO}(3, \mathbf{R})$ is such that $O^t A O$ is a diagonal matrix, then the imaginary part B of H, taken into its canonical form $O^t B O$, has only two possible non-zero entries, which are $\pm\sqrt{k}$. In particular, if $k = 0$, then $H = A$.

Proof. Let us consider a matrix $H \in P(2)$ and decompose it into its real and imaginary parts: $H = A + iB$. Then one has $\overline{H}^t = A^t - iB^t$. Also $H = \overline{H}^t$ because H is Hermitian. Hence $A = A^t$ and $B = -B^t$, i.e. A is symmetric and B is anti-symmetric. Thus the trace of A is 1, proving statement (i). One also has

$$H^2 = A^2 - B^2 + i(AB + BA),$$

and $H^2 = H$ because H is in $P(2)$. Therefore $A = A^2 - B^2$ and $B = AB + BA$.

Now, A is symmetric, and so is A^2; these two matrices obviously commute, so they can be diagonalized simultaneously by a matrix $O \in \mathrm{SO}(3, \mathbf{R})$. Since $B^2 = A^2 - A$, one knows that $O^t B^2 O$ is also diagonal:

$$O^t B^2 O = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix},$$

with $\mu_i = \lambda_i^2 - \lambda_i$, for each $i = 1, 2, 3$, where the λ_i are the eigenvalues of A. But B is antisymmetric and commutes with B^2, which is symmetric. Hence the same matrix O takes B to its canonical form:

$$O^t B O = \begin{pmatrix} 0 & a & c \\ -a & 0 & b \\ -c & -b & 0 \end{pmatrix}$$

for some $a, b, c \in \mathbf{C}$. This implies that

$$O^tB^2O = (O^tBO)(O^tBO) = \begin{pmatrix} -a^2-c^2 & -bc & ab \\ -bc & -a^2-b^2 & -ac \\ ab & -ac & -b^2-c^2 \end{pmatrix},$$

which we know is a diagonal matrix. Therefore two of the numbers a, b, c must be zero. Assume for instance that a and b are 0, then both eigenvalues λ_1 and λ_3 are roots of the polynomial

$$\lambda^2 - \lambda + c^2 = 0.$$

This implies that

$$\lambda_1 + \lambda_3 = 1 \quad \text{and} \quad \lambda_1 \cdot \lambda_3 = c^2 \geq 0.$$

Hence $\lambda_2 = 0$ (because the trace of A is 1), so 0 is an eigenvalue of A. The other eigenvalues λ_1 and λ_3 must both be ≥ 0 and ≤ 1, because their product is non-negative and their sum is 1. Moreover the roots must be real, therefore $k = c^2 \leq \frac{1}{4}$, proving statement (ii).

Also, in this case the eigenvalues of A determine the imaginary part B of H up to sign:

$$B = \pm O \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ -c & 0 & 0 \end{pmatrix} O^t,$$

with $c^2 = \lambda_1 - \lambda_3^2 = \lambda_3 - \lambda_3^2$, proving in this case the last statement of Lemma 3.5. The other cases, when either $a = c = 0$ or $b = c = 0$, are similar to the previous one. This proves that if $A = \Re(H)$ for some matrix $H \in P(2)$, then A is as stated in Lemma 3.5. Conversely, given A satisfying these conditions, the above arguments tell us how to construct B so that these matrices are the real and imaginary parts of some H in $P(2)$. $\square$

Now, given $H \in P(2)$, its real part is $\Re(H) = \frac{1}{2}(H + \overline{H})$. Define

$$\psi\colon P(2) \to M(3, \mathbf{R}),$$

the space $M(3, \mathbf{R})$ being the space of real (3×3)-matrices, by the formula

$$\psi(H) = \frac{1}{3} I_3 - \Re(H) \in M(3, \mathbf{R}), \tag{3.6}$$

where I_3 is the (3×3)-identity matrix. In other words, $\psi(H)$ is the real part of the matrix $(\frac{1}{3}I_3 - H)$. Since $H \in P(2)$, it follows that $\psi(H)$ is actually contained in $\mathcal{S}$.

It is clear that the above action of $\mathrm{SO}(3, \mathbf{R})$ on $P(2)$ given by conjugation is equivalent, via the above diffeomorphism $P(2) \cong P^2_{\mathbf{C}}$, with the standard action

studied in §2 and §3 above. It is also clear that, for every $O \in \mathrm{SO}(3, \mathbf{R})$, one has

$$\psi(O^t H O) = \frac{1}{3} I - \frac{1}{2}\big(O^t(H + \bar{H})O\big) = O^t\big(\frac{1}{3} I - \frac{1}{2}(H + \bar{H})\big)O = O^t \psi(H) O .$$

Hence we have

LEMMA 3.7. *The map ψ is equivariant. That is, for every $O \in \mathrm{SO}(3, \mathbf{R})$ and $H \in P(2)$, one has $\psi(O^t H O) = O^t \psi(H) O$.*

LEMMA 3.8. *Given $S \in \mathcal{S} - \{0\}$, there exists a unique positive $t \in \mathbf{R}$, such that the matrix $(\frac{1}{3} I - tS)$ is the real part of some matrix $H \in P(2)$.*

Proof. By Lemma 3.7, we may assume that S is diagonal. Hence the matrix $\widehat{S}_t = (\frac{1}{3} I - tS)$ is also diagonal, say

$$\widehat{S}_t = \begin{pmatrix} \lambda_1(t) & 0 & 0 \\ 0 & \lambda_2(t) & 0 \\ 0 & 0 & \lambda_3(t) \end{pmatrix}$$

with $\lambda_i(t) = \frac{1}{3} - t\mu_i$, where the μ_i are the eigenvalues of S. We notice that for all $t \in \mathbf{R}$, one has

$$\operatorname{trace} \widehat{S}_t = 1 - t\,(\operatorname{trace} S) = 1 ,$$

because S has trace 0. Hence all these matrices satisfy condition (i) of Lemma 3.5.

Let us look for the possible values of t that give solutions of Lemma 3.5. That is, we want $t > 0$ for which one eigenvalue $\lambda_i(t)$ is 0 and the others are such that their sum is 1 and their product is ≥ 0 and $\leq \frac{1}{4}$.

Let us number the eigenvalues of S so that $\mu_1 \leq \mu_2 \leq \mu_3$. Since their sum is 0 and S is not the zero matrix, one must have $\mu_1 < 0$ and $\mu_3 > 0$. If we want t as above, one $\lambda_i(t)$ must vanish. Let us look for solutions with $\lambda_1(t) = 0$. This means that $t = \frac{1}{3\mu_1} < 0$, and we want $t > 0$. Hence, there are no solutions with $\lambda_1(t) = 0$.

Now let us look for solutions with $\lambda_2(t) = 0$. This implies that $t = \frac{1}{3\mu_2}$; for this to be possible we must have $\mu_2 \neq 0$. If $\mu_2 < 0$, then $t < 0$ and we want t to be positive. Thus, we only care about $\mu_2 > 0$. We have

$$\lambda_1(t) = \frac{1}{3}(1 - \frac{\mu_1}{\mu_2}) \quad \text{and} \quad \lambda_3(t) = \frac{1}{3}(1 - \frac{\mu_3}{\mu_2}) .$$

We have $\mu_1 < 0 < \mu_2$, so $\lambda_1(t) > 0$. If $\mu_2 < \mu_3$, then $\lambda_3(t) < 0$, thus the product $\lambda_1(t)\lambda_3(t)$ is < 0, so there are no such solutions to Lemma 3.8. The

other possibility is $\mu_2 = \mu_3$; this also implies $\lambda_3(t) = 0$. In this case one has $\lambda_1(t) = 1$ and $\lambda_2(t) = \lambda_3(t) = 0$, and $t = \frac{1}{3\mu_2}$ is positive. Hence we have a solution, and this is unique because $\mu_2 = \mu_3$. If $\mu_2 = 0$, then $\lambda_2(t)$ cannot be 0 and we cannot find solutions like this.

Summarizing, so far we have seen that: i) there are no solutions as in Lemma 3.8 for which $\lambda_1(t) = 0$; ii) if $\mu_2 \leq 0$, there are no solutions as in Lemma 3.8 for which $\lambda_2(t) = 0$; and iii) if $\mu_2 = \mu_3$, then there is a unique solution as in Lemma 3.8, for which $\lambda_2(t) = \lambda_3(t) = 0$ and $\lambda_1(t) = 1$.

Finally, let us look for solutions with $\lambda_3(t) = 0$, i.e. with $t = \frac{1}{3\mu_3}$. We know, by hypothesis, that $\mu_2 \leq \mu_3$ and $\mu_3 > 0$. If $\mu_2 = \mu_3$, then we are in the previous case and there is a unique positive t giving a solution as in Lemma 3.8. Let us assume now that $\mu_2 < \mu_3$. Then we have

$$\lambda_1(t) = \frac{1}{3}(1 - \frac{\mu_1}{\mu_3}) \quad \text{and} \quad \lambda_2(t) = \frac{1}{3}(1 - \frac{\mu_2}{\mu_3}),$$

which are both ≥ 0. Since their sum is 1, it follows that each $\lambda_i(t)$ is also ≤ 1.

The product of $\lambda_1(t)$ and $\lambda_2(t)$ satisfies

$$\begin{aligned} 0 \leq \lambda_1(t) \cdot \lambda_2(t) &= \frac{1}{9}(1 - \frac{\mu_1 + \mu_2}{\mu_3} + \frac{\mu_1\mu_2}{\mu_3^2}) = \frac{1}{9}(2 + \frac{\mu_1\mu_2}{\mu_3^2}) \\ &= \frac{1}{9}(2 + \frac{\mu_1\mu_2}{(\mu_1 + \mu_2)^2}) \leq \frac{1}{4}, \end{aligned}$$

since $\mu_1 + \mu_2 + \mu_3 = 0$ and $\frac{\mu_1\mu_2}{(\mu_1+\mu_2)^2} \leq \frac{1}{4}$ because $\frac{1}{4}(a+b)^2 \geq ab$ for any real numbers a and b (with equality if and only if $a = b$). Hence $t = \frac{1}{3\mu_3}$ is the unique solution satisfying the conditions of Lemma 3.8. $\square$

We now "normalize" the map ψ so that its image is contained in $S^4 \subset \mathcal{S}$. For this we define a function

$$\alpha(H) = [\mathrm{trace}(\psi(H)^2)]^{-\frac{1}{2}},$$

i.e. $\alpha(H)$ is the inverse of the norm of $\psi(H)$ in $\mathcal{S}$, and we set

$$\Phi(H) = \alpha(H)\,\psi(H).$$

One has

$$\begin{aligned} \mathrm{trace}[\psi(H)^2] &= \mathrm{trace}[(\frac{1}{3}I_3 - \frac{1}{2}(H + \overline{H}))^2] \\ &= \mathrm{trace}[\frac{1}{9}I_3 - \frac{1}{3}(H + \overline{H}) + \frac{1}{4}(H^2 + \overline{H}^2 + H\overline{H} + \overline{H}H)] \\ &= \frac{1}{6} + \frac{1}{4}\mathrm{trace}(H\overline{H} + \overline{H}H), \end{aligned}$$

which is always positive since the matrix $(H\overline{H}+\overline{H}H)$ is positive semi-definite, so its trace is ≥ 0. Hence the maps α and Φ are well defined. It is clear that the image of Φ is contained in $S^4 \subset \mathcal{S}$, because the linearity of the trace implies that

$$[\mathrm{trace}(\Phi(H))]^2 = \alpha^2(H)\,[\mathrm{trace}\,\psi(H)]^2 = 1\,.$$

It is also clear that Φ is $\mathrm{SO}(3,\mathbf{R})$-equivariant, since the trace is invariant under conjugation and ψ is equivariant by Lemma 3.7. These considerations imply both Lemma 3.8 and the following

LEMMA 3.9. *The map Φ is an equivariant surjection from $P(2)$ over $S^4 \subset \mathcal{S}$, and it is two-to-one, except over the image of the real matrices in $P(2)$ where it is one-to-one.*

This gives the map in Theorem 3.4 that determines an equivariant diffeomorphism between S^4 and $P^2_{\mathbf{C}}$ modulo the involution given by conjugation. To complete the proof of Theorem 3.4 we need to show that Φ is invariant under the involution of $P(2)$ that corresponds to complex conjugation in $P^2_{\mathbf{C}}$. For this we notice that if L_H is the complex line in $\mathbf{C}^3$ which is the image of $H \in P(2)$, and if $0 \neq (z_1, z_2, z_3) \in L_H$, we can associate to H the point in $P^2_{\mathbf{C}}$ with projective coordinates $[z_1, z_2, z_3]$. To the matrix $\overline{H}$ there corresponds the line with projective coordinates $[\overline{z}_1, \overline{z}_2, \overline{z}_3]$. Therefore we have

LEMMA 3.10. *The involution $j*$ of $P(2)$ defined by $j*(H) = \overline{H}$ coincides with the involution j of $P^2_{\mathbf{C}}$ given by complex conjugation, $[z_1, z_2, z_3] \overset{j}{\mapsto} [\overline{z}_1, \overline{z}_2, \overline{z}_3]$.*

Then Φ is invariant under this involution, since $\Re(H) = \Re(\overline{H})$, proving Theorem 3.4. $\square$

4. SOME APPLICATIONS AND REMARKS

It is interesting to describe explicitly the orbits of the Γ action of $\mathrm{SO}(3,\mathbf{R})$ on S^4, regarded[2]) as the set of matrices with norm 1 in $\mathcal{S}$. In fact, the orbits of this action are conjugacy classes (or *congruency classes*) of traceless symmetric matrices whose square has trace 1. This is the connection between our construction and the spherical Tits buildings. Every $S \in \mathcal{S}$ can

[2]) This orbit description of S^4 is also given in [Ma2].