Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 49 (2003)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: SYMPLECTIC LOOK AT SURFACES OF REVOLUTION
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Kapitel: 1. Introduction
DOI: https://doi.org/10.5169/seals-66685

L'Enseignement Mathématique, t. 49 (2003), p. 157–172

A SYMPLECTIC LOOK AT SURFACES OF REVOLUTION

by Andrew D. HWANG

Dedicated to Professor Shoshichi Kobayashi on his 70^{th} *birthday*

1. INTRODUCTION

Informally, a surface of revolution is a 2-dimensional Riemannian manifold Σ equipped with an isometric circle action. Surfaces of revolution are among the simplest objects in differential geometry; the metric is determined by a single function of one real variable, hence can be specified by solving an *ordinary* differential equation.

A function $x\colon \Sigma \to \mathbf{R}$ is an "orbit parameter" if each level set of x is a single orbit. Given an orbit parameter, a "profile" for Σ is a function that determines the lengths of the orbits. For example, when the graph of a function ξ is revolved about an axis in $\mathbf{R}^3$, the "obvious" orbit parameter is a Cartesian coordinate x along the axis of revolution, and ξ itself is a profile.

This note constructs surfaces of revolution from an elementary, intrinsic orbit parameter and profile function. The point of departure is a theorem of Archimedes, whose proof is nowadays an easy calculus exercise. Let $S \subset \mathbf{R}^3$ be the unit sphere, regarded as a surface of revolution by fixing an arbitrary diameter. A "zone" of S is a subset bounded by two planes perpendicular to the diameter, and the "height" of a zone is the distance between its bounding planes.

THEOREM 1.1. *A zone of height h on the unit sphere has area $2\pi h$; in particular, the area depends only on the height of the zone.*

To reformulate, let x be a Cartesian coordinate along the diameter, and let O be the equator $\{x = 0\}$. Each $p \in S$ lies on a unique circle O_p perpendicular to the diameter; let $2\pi\tau$ be the oriented area bounded by O and O_p. Theorem 1.1 asserts that the extrinsic position x and the intrinsic coordinate τ are *the same orbit parameter*.

Of course, distance along the axis of rotation does not correspond so nicely with zonal area on a general surface of revolution, but in some ways area is a "better" parameter: with a judicious choice of profile function, the Gaussian curvature becomes extremely simple. The resulting description makes it easy to study and classify surfaces of revolution that have specified Gaussian curvature. The motivation for this description comes from symplectic and Kähler geometry, but the idea and methods are elementary.

2. Abstract surfaces of revolution

Identify the circle S^1 with the multiplicative group of complex numbers of norm 1, and let $\mathbf{P}^1 = \mathbf{C} \cup \{\infty\}$ be the Riemann sphere, equipped with the S^1-action induced by multiplication on $\mathbf{C}$. In this note, an *abstract surface of revolution* is a pair $\Sigma = (D, g)$ consisting of a connected, S^1-invariant domain $D \subset \mathbf{P}^1$ and an S^1-invariant metric g, possibly with conical singularities at the fixed points.

General metrics in coordinates

There are two "natural" coordinate systems on an abstract surface of revolution: *isothermal parameters* adapted to the circle action, and *action-angle coordinates*. While each highlights aspects of the metric geometry, their interplay is synergistic, and naturally suggests the "correct" choice of profile.

Isothermal parameters. A coordinate system (x, y) is said to be *isothermal* for the metric g if there exists a (locally defined) function $\psi = \psi(x, y)$ such that

$$g = e^{\psi}(dx^2 + dy^2).$$

On a surface of revolution, existence of isothermal parameters is elementary. To wit, choose local coordinates (r, θ) in which $\frac{\partial}{\partial\theta}$ generates the S^1 action. Because the metric is invariant under the circle action, the components of g