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Autor: ROEGNER, Katherine

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THE FANNING METHOD FOR CONSTRUCTING EVEN UNIMODULAR LATTICES. I

by Katherine ROEGNER

ABSTRACT. This paper provides a formal study of isofans and discusses their use in the theory of even unimodular lattices. Examples are given that illustrate how isofans simplify the construction of certain types of even unimodular lattices. A classification of isofans concludes the paper.

INTRODUCTION

The history of even unimodular lattices dates back to the 19th century when H. J. S. Smith [Sm] showed the existence of what is known today as the E_8 lattice. The even unimodular lattices have been classified for dimensions 8 [M], 16 [W2], and 24 [N]. The next dimension of interest is 32 due to the fact that even unimodular lattices only occur in dimensions divisible by 8; see e.g. [Sch]. In dimension 32, there are millions of nonisometric even unimodular lattices. Although no classification in this dimension is available, there has been considerable progress. Conway and Pless [CP] determined the doubly-even self-dual binary codes, the results of which can be transformed into a classification statement for even unimodular lattices with complete root systems of a particular type. Within their work, they noted that it is possible to build some codes using known codes by making appropriate substitutions. Kervaire [Ke] classified the remaining cases of complete even unimodular lattices in dimension 32 using a lengthy elimination procedure and a lot of machine testing. Venkov [V] has shown that, except for 15 cases, the even unimodular lattices in dimension 32 can be generated by the roots and vectors with scalar square 4. In that article, Venkov introduced an important operation on lattices, which he called "fanning". It turns out that Venkov's fanning method is comparable to Conway and Pless' substitution method.

The purpose of this article is to provide an indepth study of the fanning method. To do so, Venkov's fanning method is generalized to the isofan, a special isomorphism between rational bilinear form modules associated to root lattices. Some examples are given illustrating the construction of new complete even unimodular lattices from already known ones using isofans. In particular, an easy construction for a lattice that Conway and Pless found using "several processes including divination" is given. A classification of isofans concludes the paper.

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1. Lattices

Let $\mathbf{R}^n$ be n-dimensional euclidean space equipped with the standard scalar product

$$x \cdot y = \sum_{i=1}^{n} x_i y_i \text{ for all } x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n) \in \mathbf{R}^n .$$

A free $\mathbf{Z}$-module $\Lambda \subset \mathbf{R}^n$ of rank $k := \dim_{\mathbf{R}} \mathbf{R} \otimes_{\mathbf{Z}} \Lambda$ is called a *lattice of rank k*. A *basis* of a rank k lattice Λ is a subset $\{\lambda_1, \ldots, \lambda_k\} \subset \Lambda$ that generates Λ over $\mathbf{Z}$.

Let $\Lambda \subset \mathbf{R}^n$ be a lattice. Λ is said to be *integral* if $\lambda_i \cdot \lambda_j \in \mathbf{Z}$ for all λ_i, $\lambda_j \in \Lambda$. It is an *even* lattice if, in addition to being integral, $\lambda^2 := \lambda \cdot \lambda \in 2\mathbf{Z}$ for all $\lambda \in \Lambda$. Let $\Lambda^{\#} = \{x \in \mathbf{R}^n \mid x \cdot \lambda \in \mathbf{Z} \text{ for all } \lambda \in \Lambda\}$ denote the *dual lattice*. Clearly, Λ is integral if and only if $\Lambda \subseteq \Lambda^{\#}$. Λ is called *unimodular* if in fact $\Lambda = \Lambda^{\#}$. Thus, an even unimodular lattice is a self-dual lattice such that $\lambda^2 \in 2\mathbf{Z}$ for all $\lambda \in \Lambda$.

Let $\Lambda_1, \ldots, \Lambda_m$ be nontrivial sublattices of the integral lattice Λ whose direct sum is equal to Λ. If $x \cdot y = 0$ for all $x \in \Lambda_i$, $y \in \Lambda_j, i \neq j$, then Λ is called the orthogonal direct sum of the sublattices $\Lambda_1, \ldots, \Lambda_m$ and denoted by $\Lambda = \Lambda_1 \oplus \cdots \oplus \Lambda_m$. Λ is called *decomposable* if there exists such an orthogonal direct sum with $m > 1$, otherwise Λ is said to be *indecomposable.*

The *root system* of an even lattice Λ is the set

$$\Lambda_{\mathrm{rt}} := \{\lambda \in \Lambda \mid \lambda^2 = 2\},$$

the elements of which are called *roots*. Λ is called a *root lattice* if Λ is