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smooth model is the icosahedron.

This paper is organised as follows. In the first section I recall the results on degenerations of K3-surfaces, in particular that one can always realise a particularly nice model, the (-1)-form. Section 2 brings as illustration detailed computations for tetrahedra. The results fit in with the general deformation theory, which is treated in the third section, with special emphasis on degenerations in (-1)-form. A short fourth section introduces the combinatorial tools to handle large systems of equations: the definitions of Stanley-Reisner rings and Hodge algebras are reviewed. The final section contains the dodecahedral degenerations.

# 1. Semistable degenerations of K3-surfaces

- 1.1. The name K3 has been explained by André Weil: «en l'honneur de Kummer, Kähler, Kodaira et de la belle montagne K2 au Cachemire» [W, p. 546]. He calls any surface a K3, if it has the differentiable structure of a smooth quartic surface in  $\mathbf{P}^3(\mathbf{C})$ . A Kummer surface is a quartic with 16  $A_1$ -singularities. As these singularities admit simultaneous resolution, the minimal resolution of a Kummer surface deforms into a smooth quartic and is therefore a K3-surface. A quartic surface X is simply connected, so in particular  $b_1(X) = 0$  and has trivial canonical sheaf by the adjunction formula: X is an anti-canonical divisor in  $\mathbf{P}^3$ . The modern definition of a K3-surface:  $b_1(X) = 0$  and  $K_X = 0$ , is equivalent with Weil's definition because all K3-surfaces form one connected family.
- 1.2. Let  $f: \mathcal{X} \to S \ni 0$  be a proper surjective holomorphic map of a 3-dimensional complex manifold  $\mathcal{X}$  to a (germ of a) curve S such that the zero fibre  $X = f^{-1}(0)$  is a reduced divisor with (simple) normal crossings; then the degeneration f is called *semistable*.

In the K3 case the following holds (see [F-M] for exact references):

1.3. THEOREM (Kulikov). Let  $f: \mathcal{X} \to S$  be a semistable degeneration of K3-surfaces. If all components of  $X = f^{-1}(0)$  are Kähler, then there exists a modification  $\mathcal{X}'$  of  $\mathcal{X}$  such that  $K_{\mathcal{X}'} \equiv 0$ .

A degeneration as in the conclusion of the theorem  $(K_{\mathcal{X}} \equiv 0)$  is called a *Kulikov model*.

- 1.4. THEOREM (Persson, Kulikov). Let  $f: \mathcal{X} \to S$  be a Kulikov model of a degeneration of K3-surfaces with all components of  $X = f^{-1}(0)$  Kähler. Then either
  - (I) X is smooth, or
  - (II) X is a chain of elliptic ruled components with rational surfaces at the ends and all double curves are smooth elliptic curves, or
- (III) X consists of rational surfaces meeting along rational curves which form cycles on each component. The dual graph is a triangulation of  $S^2$ .

According to the case division in the theorem one speaks of degenerations of type I, II, or III. Without the Kähler assumption it is not always possible to arrange that  $K_{\mathcal{X}} \equiv 0$  [K, N]. Even under the assumption  $K_{\mathcal{X}} \equiv 0$  the list becomes longer (see [N, Thm. 2.1]). In particular it is possible that the central fibre contains surfaces of type VII<sub>0</sub>. The case that the central fibre contains an Inoue-Hirzebruch surface is relevant for the deformation of cusp singularities [L].

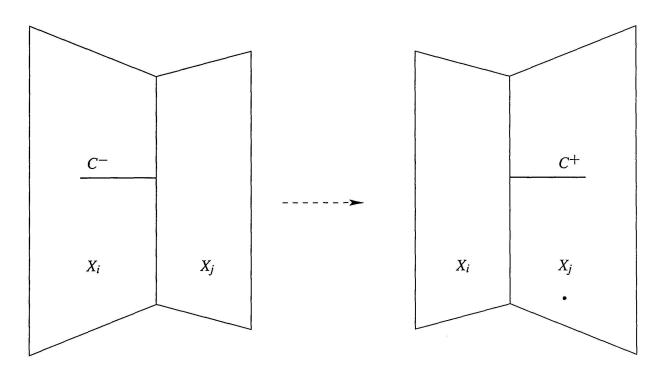


FIGURE 1.1 Elementary modification of type I

A Kulikov model is not unique. The central fibre can be modified with flops. If  $C^-$  is a smooth rational curve in X with self intersection -1, lying in a component  $X_i$  and intersecting the double curve transversally in one point lying in  $X_j$ , then after the flop the curve  $C^+$  lies in  $X_j$ . This operation is

also called an elementary modification of type I along  $C^-$ . An elementary modification of type II is a flop in a curve  $C^-$ , which is a component of the double curve and has self intersection -1 on both components  $X_i$ ,  $X_j$  on which it lies. There are two triple points on  $C^-$  involving the components  $X_k$  and  $X_l$ . After the flop  $C^+$  is a double curve lying in the components  $X_k$  and  $X_l$ . Note that we might lose projectivity by using elementary transformations.

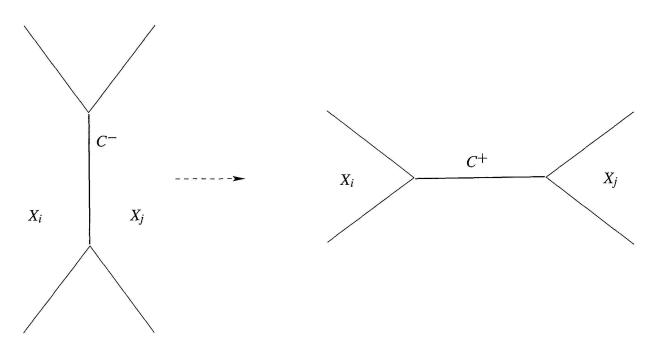


FIGURE 1.2 Elementary modification of type II

- 1.5. THE MINUS ONE THEOREM [M-M]. By modifications of type I and II one can achieve that every component of the double curve of the special fibre has self intersection -1 on both components on which it lies.
- 1.6. Let  $X = \bigcup X_i$  be a normal crossings surface with double locus D. If X is a divisor in a smooth 3-fold M then one can define the infinitesimal normal bundle  $\mathcal{O}_D(X)$  as  $\mathcal{O}_D(X) = \mathcal{O}_M(X)|_D$ . It can be defined independently of M. To this end, let  $I_{X_i}$  be the ideal sheaf of  $X_i$  in X. It is locally generated by one generator  $z_i$ , but is not invertible as  $z_i$  is a zero divisor in  $\mathcal{O}_X$ . However  $I_{X_i}|_D$  is locally free [F2, (1.8)]. Following Friedman one makes the following definitions.
- 1.7. DEFINITION. The infinitesimal normal bundle  $\mathcal{O}_D(X)$  is the line bundle dual to  $\mathcal{O}_D(-X)$ , where

$$\mathcal{O}_D(-X) = (I_{X_1}|_D) \otimes_{\mathcal{O}_D} \cdots \otimes_{\mathcal{O}_D} (I_{X_k}|_D).$$

If X is a divisor in M the bundle thus defined is equal to  $\mathcal{O}_M(X)|_D$ . In particular, if X is a central fibre in a semistable degeneration  $\mathcal{X} \to S$ , then  $\mathcal{O}_{\mathcal{X}}(X) \equiv \mathcal{O}_{\mathcal{X}}$  so  $\mathcal{O}_D(X) = \mathcal{O}_D$ . This gives a necessary condition for being a central fibre.

1.8. DEFINITION. The normal crossings surface X is d-semistable if  $\mathcal{O}_D(X) = \mathcal{O}_D$ .

A consequence is the triple point formula: let  $D_{ij} = X_i \cap X_j$  and denote by  $(D_{ij})_{X_i}^2$  the self intersection of  $D_{ij}$  on  $X_i$  and by  $T_{ij}$  the number of triple points on  $D_{ij}$ . Then (cf. [P, Cor. 2.4.2])

$$(D_{ij})_{X_i}^2 + (D_{ij})_{X_j}^2 + T_{ij} = 0.$$

1.9. DEFINITION. A compact normal crossings surface is a d-semistable K3-surface of type III if X is d-semistable,  $\omega_X = \mathcal{O}_X$  and each  $X_i$  is rational, the double curves  $D_i \subset X_i$  are cycles of rational curves and the dual graph triangulates  $S^2$ . If the conclusions of the Minus One Theorem hold, that every component of the double curve has self intersection -1 on either component of X on which it lies, the surface X is said to be in (-1)-form.

## 2. Tetrahedra

2.1. To realise a tetrahedron we start out with four general planes in 3-space. They do not form a d-semistable K3-surface, but the dual graph is a tetrahedron. To write down a degeneration with this special fibre we just take the pencil spanned by  $T = x_0x_1x_2x_3$  and a smooth quartic. The symmetry group of the tetrahedron (including reflections) acts if we only take  $S_4$ -invariant quartics:

$$Q = a\sigma_1^4 + b\sigma_1^2\sigma_2 + c\sigma_2^2 + d\sigma_1\sigma_3,$$

where the  $\sigma_i$  are the elementary symmetric functions in the four variables  $x_i$  and a, b, c and d are constants.

To obtain a family  $f: \mathcal{X} \to S$  one has to blow up the base locus of the pencil. This can be done in several ways. Blowing up T = Q = 0 gives a total space which is singular, with in general 24 ordinary double points coming from the 24 intersection points of Q with the double curve of the tetrahedron T. Arguably, this is the nicest model, and the best one can hope for in view of the theory of minimal models of 3-folds. A smooth model is