Zeitschrift:	L'Enseignement Mathématique
Herausgeber:	Commission Internationale de l'Enseignement Mathématique
Band:	48 (2002)
Heft:	1-2: L'ENSEIGNEMENT MATHÉMATIQUE
Artikel:	SEMISTABLE K3-SURFACES WITH ICOSAHEDRAL SYMMETRY
Autor:	Stevens, Jan
Kapitel:	1. Semistable degenerations of K3 -surfaces
DOI:	https://doi.org/10.5169/seals-66069

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. <u>Mehr erfahren</u>

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. <u>En savoir plus</u>

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. <u>Find out more</u>

Download PDF: 06.08.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

smooth model is the icosahedron.

This paper is organised as follows. In the first section I recall the results on degenerations of K3-surfaces, in particular that one can always realise a particularly nice model, the (-1)-form. Section 2 brings as illustration detailed computations for tetrahedra. The results fit in with the general deformation theory, which is treated in the third section, with special emphasis on degenerations in (-1)-form. A short fourth section introduces the combinatorial tools to handle large systems of equations: the definitions of Stanley-Reisner rings and Hodge algebras are reviewed. The final section contains the dodecahedral degenerations.

1. Semistable degenerations of K3-surfaces

1.1. The name K3 has been explained by André Weil: «en l'honneur de Kummer, Kähler, Kodaira et de la belle montagne K2 au Cachemire» [W, p. 546]. He calls any surface a K3, if it has the differentiable structure of a smooth quartic surface in $\mathbf{P}^3(\mathbf{C})$. A Kummer surface is a quartic with 16 A_1 -singularities. As these singularities admit simultaneous resolution, the minimal resolution of a Kummer surface deforms into a smooth quartic and is therefore a K3-surface. A quartic surface X is simply connected, so in particular $b_1(X) = 0$ and has trivial canonical sheaf by the adjunction formula: X is an anti-canonical divisor in \mathbf{P}^3 . The modern definition of a K3-surface: $b_1(X) = 0$ and $K_X = 0$, is equivalent with Weil's definition because all K3-surfaces form one connected family.

1.2. Let $f: \mathcal{X} \to S \ni 0$ be a proper surjective holomorphic map of a 3-dimensional complex manifold \mathcal{X} to a (germ of a) curve S such that the zero fibre $X = f^{-1}(0)$ is a reduced divisor with (simple) normal crossings; then the degeneration f is called *semistable*.

In the K3 case the following holds (see [F-M] for exact references):

1.3. THEOREM (Kulikov). Let $f: \mathcal{X} \to S$ be a semistable degeneration of K3-surfaces. If all components of $X = f^{-1}(0)$ are Kähler, then there exists a modification \mathcal{X}' of \mathcal{X} such that $K_{\mathcal{X}'} \equiv 0$.

A degeneration as in the conclusion of the theorem $(K_{\mathcal{X}} \equiv 0)$ is called a *Kulikov model*.

1.4. THEOREM (Persson, Kulikov). Let $f: \mathcal{X} \to S$ be a Kulikov model of a degeneration of K3-surfaces with all components of $X = f^{-1}(0)$ Kähler. Then either

(I) X is smooth, or

- (II) X is a chain of elliptic ruled components with rational surfaces at the ends and all double curves are smooth elliptic curves, or
- (III) X consists of rational surfaces meeting along rational curves which form cycles on each component. The dual graph is a triangulation of S^2 .

According to the case division in the theorem one speaks of degenerations of type I, II, or III. Without the Kähler assumption it is not always possible to arrange that $K_{\chi} \equiv 0$ [K, N]. Even under the assumption $K_{\chi} \equiv 0$ the list becomes longer (see [N, Thm. 2.1]). In particular it is possible that the central fibre contains surfaces of type VII₀. The case that the central fibre contains an Inoue-Hirzebruch surface is relevant for the deformation of cusp singularities [L].



FIGURE 1.1 Elementary modification of type I

A Kulikov model is not unique. The central fibre can be modified with flops. If C^- is a smooth rational curve in X with self intersection -1, lying in a component X_i and intersecting the double curve transversally in one point lying in X_j , then after the flop the curve C^+ lies in X_j . This operation is

SEMISTABLE K3-SURFACES WITH ICOSAHEDRAL SYMMETRY

also called an elementary modification of type I along C^- . An elementary modification of type II is a flop in a curve C^- , which is a component of the double curve and has self intersection -1 on both components X_i , X_j on which it lies. There are two triple points on C^- involving the components X_k and X_l . After the flop C^+ is a double curve lying in the components X_k and X_l . Note that we might lose projectivity by using elementary transformations.



FIGURE 1.2 Elementary modification of type II

1.5. THE MINUS ONE THEOREM [M-M]. By modifications of type I and II one can achieve that every component of the double curve of the special fibre has self intersection -1 on both components on which it lies.

1.6. Let $X = \bigcup X_i$ be a normal crossings surface with double locus D. If X is a divisor in a smooth 3-fold M then one can define the infinitesimal normal bundle $\mathcal{O}_D(X)$ as $\mathcal{O}_D(X) = \mathcal{O}_M(X)|_D$. It can be defined independently of M. To this end, let I_{X_i} be the ideal sheaf of X_i in X. It is locally generated by one generator z_i , but is not invertible as z_i is a zero divisor in \mathcal{O}_X . However $I_{X_i}|_D$ is locally free [F2, (1.8)]. Following Friedman one makes the following definitions.

1.7. DEFINITION. The *infinitesimal normal bundle* $\mathcal{O}_D(X)$ is the line bundle dual to $\mathcal{O}_D(-X)$, where

$$\mathcal{O}_D(-X) = (I_{X_1}|_D) \otimes_{\mathcal{O}_D} \cdots \otimes_{\mathcal{O}_D} (I_{X_k}|_D).$$

J. STEVENS

If X is a divisor in M the bundle thus defined is equal to $\mathcal{O}_M(X)|_D$. In particular, if X is a central fibre in a semistable degeneration $\mathcal{X} \to S$, then $\mathcal{O}_{\mathcal{X}}(X) \equiv \mathcal{O}_{\mathcal{X}}$ so $\mathcal{O}_D(X) = \mathcal{O}_D$. This gives a necessary condition for being a central fibre.

1.8. DEFINITION. The normal crossings surface X is *d*-semistable if $\mathcal{O}_D(X) = \mathcal{O}_D$.

A consequence is the triple point formula: let $D_{ij} = X_i \cap X_j$ and denote by $(D_{ij})_{X_i}^2$ the self intersection of D_{ij} on X_i and by T_{ij} the number of triple points on D_{ij} . Then (cf. [P, Cor. 2.4.2])

$$(D_{ij})_{X_i}^2 + (D_{ij})_{X_i}^2 + T_{ij} = 0.$$

1.9. DEFINITION. A compact normal crossings surface is a *d*-semistable K3-surface of type III if X is *d*-semistable, $\omega_X = \mathcal{O}_X$ and each X_i is rational, the double curves $D_i \subset X_i$ are cycles of rational curves and the dual graph triangulates S^2 . If the conclusions of the Minus One Theorem hold, that every component of the double curve has self intersection -1 on either component of X on which it lies, the surface X is said to be in (-1)-form.

2. Tetrahedra

2.1. To realise a tetrahedron we start out with four general planes in 3-space. They do not form a *d*-semistable K3-surface, but the dual graph is a tetrahedron. To write down a degeneration with this special fibre we just take the pencil spanned by $T = x_0 x_1 x_2 x_3$ and a smooth quartic. The symmetry group of the tetrahedron (including reflections) acts if we only take S_4 -invariant quartics:

$$Q = a\sigma_1^4 + b\sigma_1^2\sigma_2 + c\sigma_2^2 + d\sigma_1\sigma_3 \,,$$

where the σ_i are the elementary symmetric functions in the four variables x_i and a, b, c and d are constants.

To obtain a family $f: \mathcal{X} \to S$ one has to blow up the base locus of the pencil. This can be done in several ways. Blowing up T = Q = 0 gives a total space which is singular, with in general 24 ordinary double points coming from the 24 intersection points of Q with the double curve of the tetrahedron T. Arguably, this is the nicest model, and the best one can hope for in view of the theory of minimal models of 3-folds. A smooth model is