

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 48 (2002)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE HILBERT METRIC AND GROMOV HYPERBOLICITY
Autor: Karlsson, Anders / NOSKOV, Guennadi A.
Kapitel: 4. CONSEQUENCES OF GROMOV HYPERBOLICITY FOR THE SHAPE OF THE BOUNDARY
DOI: <https://doi.org/10.5169/seals-66068>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 17.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

is bounded in view of the estimate $\kappa^{-1}(x^2 + y^2) < f(x, y) < \kappa(x^2 + y^2)$ for some universal $\kappa > 0$ and of the fact that $f(x, y) = \varepsilon$ on C_ε . \square

4. CONSEQUENCES OF GROMOV HYPERBOLICITY FOR THE SHAPE OF THE BOUNDARY

PROPOSITION 4.1. *Let D be a bounded convex domain in \mathbf{R}^n and let h be a Hilbert metric on D . If h is Gromov hyperbolic then the boundary ∂D is strictly convex, that is, it does not contain a line segment.*

This can be proven following the proof of N. Ivanov [Iv97] of Masur-Wolf's theorem [MW95] that the Teichmüller spaces (genus ≥ 2) are not Gromov hyperbolic. The proof makes use of Gromov's exponential divergence criterion, see [BH99, p.412]. For another proof of the above proposition, see [SM00].

THEOREM 4.2. *Let D be a bounded convex domain in \mathbf{R}^n and let h be the Hilbert metric on D . If h is Gromov hyperbolic then the boundary ∂D is smooth of class C^1 .*

Proof. *2-dimensional case:* First, by the previous result, D is strictly convex. Let $y = f(x)$, $x \in (-a, a)$ be an equation of ∂D near some point. Then f is strictly convex and hence the one-sided derivatives $f'_-(x)$, $f'_+(x)$ exist and are strictly increasing on $(\varepsilon, \varepsilon)$, [RV73, §11].

We prove that $f'_-(0) = f'_+(0)$. Suppose not, then by choosing appropriate Cartesian coordinates we may assume that $f'_-(0) < 0$ and $f'_+(0) > 0$. For each sufficiently small ε construct an ideal triangle $\Delta = \Delta(\varepsilon)$ in D with one vertex 0 and two other vertices corresponding to the intersection of the line $y = \varepsilon$ with ∂D . We assert that the slimness of $\Delta(\varepsilon)$ tends to ∞ when ε tends to zero. Namely we show that the Hilbert distance between the point $P = (0, \varepsilon)$ and any point Q of the side $[0, B]$ tends to ∞ . Let $f'_+(0) = \tan \alpha$, $0 < \alpha < \pi/2$. Let $x_1 < x_2$ be the points such that $f(x_1) = \varepsilon$ and $f'_+(0)x_2 = \varepsilon$. Then

$$PQ \geq \varepsilon \cos \alpha = f(x_1) \cos \alpha.$$

Let O, R be the intersection points of the line PQ with ∂D . We have therefore

$$QR \leq x_2 - x_1 = \frac{f(x_1)}{f'_+(0)} - x_1 = \frac{f(x_1) - f'_+(0)x_1}{f'_+(0)}$$

and hence, combining the last two inequalities,

$$\begin{aligned} \frac{PQ}{QR} &\geq \frac{f'_+(0)f(x_1)\cos\alpha}{f(x_1)-f'_+(0)x_1} \\ &= \frac{f'_+(0)\cos\alpha}{1-f'_+(0)\frac{x_1}{f(x_1)}} \rightarrow \infty \text{ when } x_1 \rightarrow 0. \end{aligned}$$

It follows that

$$h(P, Q) = \ln \left(1 + \frac{PQ}{OP} \right) \left(1 + \frac{PQ}{QR} \right) \rightarrow \infty \text{ when } x_1 \rightarrow 0$$

and hence the slimness of $\Delta(\varepsilon)$ tends to ∞ when ε tends to zero.

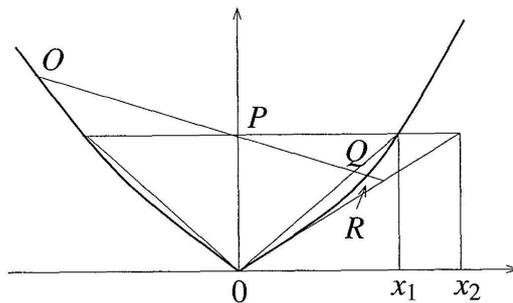


FIGURE 4

Hyperbolicity implies C^1

It remains to show that f' is continuous. By [RV73, §14] we have

$$\begin{aligned} \lim_{x \rightarrow x_0^+} f'_+(x) &= f'_+(x_0), \\ \lim_{x \rightarrow x_0^-} f'_+(x) &= f'_-(x_0). \end{aligned}$$

From this we conclude that f'_+ is continuous at x_0 since $f'_+(x_0) = f'_-(x_0)$. But $f'_-(x_0) = f'_+(x_0)$ hence f' is also continuous at x_0 .

n-dimensional case: Recall the known result that if f is a differentiable convex function defined on an open convex set S in \mathbf{R}^{n+1} , then it is C^1 on S , see for example [RV73]. Let D be a bounded convex domain in \mathbf{R}^{n+1} , $n \geq 2$. It is enough to prove that ∂D is differentiable at any point. Given a point $p \in \partial D$, we can choose the coordinate axis of \mathbf{R}^{n+1} so that the origin O of the coordinates is at p , all of D lies in the halfspace $x_0 \geq 0$ and in a neighbourhood of p the surface ∂D can be represented as the graph of a nonpositive convex function $x_0 = f(x_1, x_2, \dots, x_n)$, $x = (x_1, x_2, \dots, x_n)$, $f(0) = 0$. Considering the 2-dimensional sections in the planes x_0, x_i , $i = 1, \dots, n$, we obtain that the partial derivatives of f at 0 exist and $f_{x_i}(0) = 0$, $i = 1, \dots, n$. We have to prove that for each $\varepsilon > 0$ there is a

neighbourhood U_ε of 0 such that $f(x) < \varepsilon|x|$ in this neighbourhood. But in view of $f_{x_i}(0) = 0$, $i = 1, \dots, n$, we have $f(0, \dots, 0, x_i, 0, \dots, 0) < \varepsilon|x_i|$ for sufficiently small x_i and hence by convexity $f(x) < \varepsilon|x|$ for sufficiently small $|x|$. \square

REMARK 4.3. The following was announced in [B00]: *If a strictly convex domain D is divisible, that is, if it admits a proper cocompact group of isometries Γ , then D is Gromov hyperbolic if and only if ∂D is C^1 .* Our Theorem 4.2 shows that in the implication (Gromov hyperbolicity + divisibility $\Rightarrow C^1$) the condition of divisibility is superfluous.

5. NON-STRICTLY CONVEX DOMAINS

This section owes much of its existence to [Be97] and [Be99]. Using a different argument, we prove certain extensions to arbitrary convex bounded domains of some of the results obtained in those papers.

LEMMA 5.1. *Let D be a bounded convex domain in \mathbf{R}^n . Let $\{x_n\}, \{y_n\}$ be two sequences of points in D . Assume that $x_n \rightarrow \bar{x} \in \partial D$, $y_n \rightarrow \bar{y} \in \bar{D}$ and $[\bar{x}, \bar{y}] \not\subseteq \partial D$. Let x'_n and y'_n denote the endpoints of the chord through x_n and y_n as usual. Then x'_n converges to \bar{x} and y'_n converges to the endpoint \bar{y}' of the chord defined by \bar{x} and \bar{y} different from \bar{x} .*

Proof. Compare with Lemma 5.3. in [Be97]. Every limit point of chord endpoints must belong to the line through \bar{x} and \bar{y} . In addition, in the case of x'_n for example, any limit point must lie on the halfline from \bar{x} not containing \bar{y} . At the same time each limit point must belong to the boundary of D , and the statement follows since the line through \bar{x} and \bar{y} intersects ∂D only in \bar{x} and \bar{y}' . \square

THEOREM 5.2. *Let D be a bounded convex domain. Let $\{x_n\}$ and $\{z_n\}$ be two sequences of points in D . Assume that $x_n \rightarrow \bar{x} \in \partial D$, $z_n \rightarrow \bar{z} \in \partial D$ and $[\bar{x}, \bar{z}] \not\subseteq \partial D$. Then there is a constant $K = K(\bar{x}, \bar{z})$ such that for the Gromov product $(x_n | z_n)_y$ in Hilbert distances relative to some fixed point y in D we have*

$$\limsup_{n \rightarrow \infty} (x_n | z_n)_y \leq K.$$