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Autor: Karlsson, Anders / NOSKOV, Guennadi A.

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is bounded in view of the estimate $\kappa^{-1}(x^2+y^2) < f(x,y) < \kappa(x^2+y^2)$ for some universal $\kappa > 0$ and of the fact that $f(x,y) = \varepsilon$ on C_ε. □

4. Consequences of Gromov hyperbolicity for the shape of the boundary

PROPOSITION 4.1. *Let D be a bounded convex domain in $\mathbf{R}^n$ and let h be a Hilbert metric on D. If h is Gromov hyperbolic then the boundary ∂D is strictly convex, that is, it does not contain a line segment.*

This can be proven following the proof of N. Ivanov [Iv97] of Masur-Wolf's theorem [MW95] that the Teichmüller spaces (genus ≥ 2) are not Gromov hyperbolic. The proof makes use of Gromov's exponential divergence criterion, see [BH99, p. 412]. For another proof of the above proposition, see [SM00].

THEOREM 4.2. *Let D be a bounded convex domain in $\mathbf{R}^n$ and let h be the Hilbert metric on D. If h is Gromov hyperbolic then the boundary ∂D is smooth of class C^1.*

Proof. *2-dimensional case:* First, by the previous result, D is strictly convex. Let $y = f(x)$, $x \in (-a, a)$ be an equation of ∂D near some point. Then f is strictly convex and hence the one-sided derivatives $f'_-(x)$, $f'_+(x)$ exist and are strictly increasing on $(\varepsilon, \varepsilon)$, [RV73, § 11].

We prove that $f'_-(0) = f'_+(0)$. Suppose not, then by choosing appropriate Cartesian coordinates we may assume that $f'_-(0) < 0$ and $f'_+(0) > 0$. For each sufficiently small ε construct an ideal triangle $\Delta = \Delta(\varepsilon)$ in D with one vertex 0 and two other vertices corresponding to the intersection of the line $y = \varepsilon$ with ∂D. We assert that the slimness of $\Delta(\varepsilon)$ tends to ∞ when ε tends to zero. Namely we show that the Hilbert distance between the point $P = (0, \varepsilon)$ and any point Q of the side $[0, B]$ tends to ∞. Let $f'_+(0) = \tan\alpha$, $0 < \alpha < \pi/2$. Let $x_1 < x_2$ be the points such that $f(x_1) = \varepsilon$ and $f'_+(0)x_2 = \varepsilon$. Then

$$PQ \geq \varepsilon \cos\alpha = f(x_1)\cos\alpha .$$

Let O, R be the intersection points of the line PQ with ∂D. We have therefore

$$QR \leq x_2 - x_1 = \frac{f(x_1)}{f'_+(0)} - x_1 = \frac{f(x_1) - f'_+(0)x_1}{f'_+(0)}$$

and hence, combining the last two inequalities,

$$\frac{PQ}{QR} \geq \frac{f'_+(0) f(x_1) \cos\alpha}{f(x_1) - f'_+(0)\, x_1}$$
$$= \frac{f'_+(0) \cos\alpha}{1 - f'_+(0) \frac{x_1}{f(x_1)}} \longrightarrow \infty \text{ when } x_1 \to 0\,.$$

It follows that

$$h(P,Q) = \ln \left(1 + \frac{PQ}{OP}\right) \left(1 + \frac{PQ}{QR}\right) \longrightarrow \infty \text{ when } x_1 \to 0$$

and hence the slimness of $\Delta(\varepsilon)$ tends to ∞ when ε tends to zero.

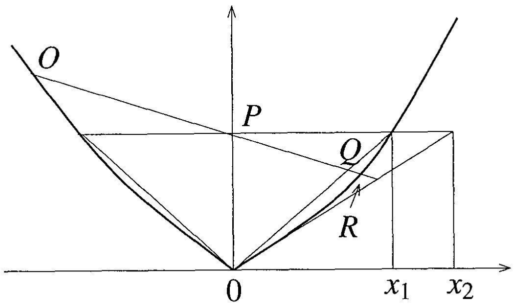

FIGURE 4
Hyperbolicity implies C^1

It remains to show that f' is continuous. By [RV73, § 14] we have

$$\lim_{x \to x_0 +} f'_+(x) = f'_+(x_0)\,,$$
$$\lim_{x \to x_0 -} f'_+(x) = f'_-(x_0)\,.$$

From this we conclude that f'_+ is continuous at x_0 since $f'_+(x_0) = f'_-(x_0)$. But $f'(x_0) = f'_+(x_0)$ hence f' is also continuous at x_0.

n-dimensional case: Recall the known result that if f is a differentiable convex function defined on an open convex set S in $\mathbf{R}^{n+1}$, then it is C^1 on S, see for example [RV73]. Let D be a bounded convex domain in $\mathbf{R}^{n+1}, n \geq 2$. It is enough to prove that ∂D is differentiable at any point. Given a point $p \in \partial D$, we can choose the coordinate axis of $\mathbf{R}^{n+1}$ so that the origin O of the coordinates is at p, all of D lies in the halfspace $x_0 \geq 0$ and in a neighbourhood of p the surface ∂D can be represented as the graph of a nonpositive convex function $x_0 = f(x_1, x_2, \ldots, x_n)$, $x = (x_1, x_2, \ldots, x_n)$, $f(0) = 0$. Considering the 2-dimensional sections in the planes x_0, x_i, $i = 1, \ldots, n$, we obtain that the partial derivatives of f at 0 exist and $f_{x_i}(0) = 0$, $i = 1, \ldots, n$. We have to prove that for each $\varepsilon > 0$ there is a neighbourhood U_ε of 0 such that $f(x) < \varepsilon|x|$ in this neighbourhood. But in view of $f_{x_i}(0) = 0,\ i = 1, \ldots, n$, we have $f(0, \ldots, 0, x_i, 0, \ldots, 0) < \varepsilon|x_i|$ for sufficiently small x_i and hence by convexity $f(x) < \varepsilon|x|$ for sufficiently small $|x|$. □

REMARK 4.3. The following was announced in [B00]: *If a strictly convex domain D is divisible, that is, if it admits a proper cocompact group of isometries Γ, then D is Gromov hyperbolic if and only if ∂D is C^1.* Our Theorem 4.2 shows that in the implication (Gromov hyperbolicity + divisibility $\Rightarrow C^1$) the condition of divisibility is superfluous.

5. NON-STRICTLY CONVEX DOMAINS

This section owes much of its existence to [Be97] and [Be99]. Using a different argument, we prove certain extensions to arbitrary convex bounded domains of some of the results obtained in those papers.

LEMMA 5.1. *Let D be a bounded convex domain in $\mathbf{R}^n$. Let $\{x_n\}, \{y_n\}$ be two sequences of points in D. Assume that $x_n \to \bar{x} \in \partial D$, $y_n \to \bar{y} \in \bar{D}$ and $[\bar{x}, \bar{y}] \not\subseteq \partial D$. Let x'_n and y'_n denote the endpoints of the chord through x_n and y_n as usual. Then x'_n converges to $\bar{x}$ and y'_n converges to the endpoint $\bar{y}'$ of the chord defined by $\bar{x}$ and $\bar{y}$ different from $\bar{x}$.*

Proof. Compare with Lemma 5.3. in [Be97]. Every limit point of chord endpoints must belong to the line through $\bar{x}$ and $\bar{y}$. In addition, in the case of x'_n for example, any limit point must lie on the halfline from $\bar{x}$ not containing $\bar{y}$. At the same time each limit point must belong to the boundary of D, and the statement follows since the line through $\bar{x}$ and $\bar{y}$ intersects ∂D only in $\bar{x}$ and $\bar{y}'$. □

THEOREM 5.2. *Let D be a bounded convex domain. Let $\{x_n\}$ and $\{z_n\}$ be two sequences of points in D. Assume that $x_n \to \bar{x} \in \partial D$, $z_n \to \bar{z} \in \partial D$ and $[\bar{x}, \bar{z}] \not\subseteq \partial D$. Then there is a constant $K = K(\bar{x}, \bar{z})$ such that for the Gromov product $(x_n \mid z_n)_y$ in Hilbert distances relative to some fixed point y in D we have*

$$\limsup_{n \to \infty} (x_n \mid z_n)_y \leq K .$$