Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 48 (2002)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE HILBERT METRIC AND GROMOV HYPERBOLICITY

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Kapitel: 1.2 Chords larger than

DOI: https://doi.org/10.5169/seals-66068

$$\frac{a_1 a_2}{b_1 b_2} = \frac{\sin\alpha_1 \sin\alpha_2}{\sin\beta_2 \sin\beta_1} = \frac{2\sin\alpha_1}{|A_1B_1|}\frac{|A_2B_2|}{2\sin\beta_2}\frac{2\sin\alpha_2}{|A_2B_2|}\frac{|A_1B_1|}{2\sin\beta_1}$$
$$= \frac{K(A_1,B_1,B_2)K(A_2,B_1,B_2)}{K(B_1,A_1,A_2)K(B_2,A_1,A_2)}. \qquad \square$$

COROLLARY 1.2. *Let D be a bounded convex domain in $\mathbf{R}^n$. Assume that there is a constant $C > 0$ such that*

$$\frac{K(x,y,z)}{K(x',y',z')} \leq C$$

for any two triples of distinct points in ∂D all lying in the same 2-dimensional plane. Then D satisfies the intersecting chords property.

Proof. Any two intersecting chords define a plane and by Proposition 1.1 we have

$$\frac{a_1 a_2}{b_1 b_2} = \frac{K_{\alpha 1}K_{\alpha 2}}{K_{\beta 1}K_{\beta 2}} \leq C^2. \qquad \square$$

REMARK 1.3. In view of this subsection it is clear that ICP implies restrictions on the curvature of the boundary, e.g. there cannot be any points of zero curvature. We were however not able to establish the converse of Corollary 1.2.

1.2 CHORDS LARGER THAN δ

The following proposition provides a different approach to the result in [Be97] mentioned above.

PROPOSITION 1.4. *Let D be a bounded convex domain in $\mathbf{R}^n$. Let δ be such that the length of any line segment contained in ∂D is bounded from above by some $\delta' < \delta$. Then there is a constant $C = C(D,\delta) > 0$ such that*

$$C(D,\delta) \leq K(x,y,z) \leq \frac{2}{\delta}, \tag{1.3}$$

whenever $x,y,z \in \partial D$ and $xy \geq \delta$.

Proof. The angle $\alpha(x,y,v) := \angle_y(xy,v)$ is continuous in $x,y \in \mathbf{R}^n$ and $v \in UT_y(\partial D)$, the unit tangent cone at y. The tangent cone at a boundary point y is the union of all hyperplanes containing y but which are disjoint from D. If $[x,y]$ does not lie in ∂D, then $0 < \alpha(x,y,v) < \pi$. The set

$$S = \{(x,y,v) \in \partial D \times \partial D \times UT_y(\partial D) \,:\, xy \geq \delta\}$$

is compact. Hence there is a constant $\alpha_0 > 0$ such that

$$\alpha_0 \leq \alpha(x, y, v) \leq \pi - \alpha_0 \tag{1.4}$$

for every $(x, y, v) \in S$. By the definition of the tangent cone and compactness there is an $\varepsilon > 0$ such that for any $y, z \in \partial D$, $0 < yz < \varepsilon$ there is an element $v \in UT_y(\partial D)$ for which

$$0 \leq \angle_y(yz, v) \leq \alpha_0/2\,. \tag{1.5}$$

The estimates (1.4) and (1.5) imply the existence of $C > 0$ and the other inequality in (1.3) is trivial. □

As an immediate consequence of Propositions 1.1 and 1.4 we have:

COROLLARY 1.5 (cf. [Be99]). *Let D be a bounded convex domain such that any line segment in ∂D has length less than $\delta' < \delta$. Then the intersecting chords property holds for any two chords each of length greater than δ.*

2. HYPERBOLICITY OF HILBERT'S METRIC

Let (Y, d) be a metric space. Given two points $z, w \in Y$, let

$$(z \mid w)_y = \frac{1}{2}(d(z, y) + d(w, y) - d(z, w))$$

be their *Gromov product* relative to y. We think of y as a fixed base point. The metric space Y is *Gromov hyperbolic* (or *δ-hyperbolic*) if there is a constant $\delta \geq 0$ such that the inequality

$$(x \mid z)_y \geq \min\{(x \mid w)_y, (w \mid z)_y\} - \delta$$

holds for any four points x, y, z, w in Y. As is known, it is enough to show such an inequality for a fixed y (the δ changes by a factor of 2); see [BH99] for a proof of this and we also refer to this book for a general exposition of this important notion of hyperbolicity. By expanding the terms the above inequality is equivalent to

$$d(x, z) + d(y, w) \leq \max\{d(x, y) + d(z, w), d(y, z) + d(x, w)\} + 2\delta\,. \tag{2.1}$$