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Artikel: THE HILBERT METRIC AND GROMOV HYPERBOLICITY
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We say that a domain satisfies the *intersecting chords property* (ICP) if (1.1) holds for *any* two intersecting chords c_1 and c_2. It is easy to see that ICP may fail for a general strictly convex domain (at a curvature zero point or a 'corner').

We show in this section that ICP holds for domains that satisfy a certain (non-differentiable) curvature condition. Domains with C^2 boundary of nonvanishing curvature are proved to satisfy this condition in Section 3.

1.1 Intersecting line segments and Menger curvature

This subsection clarifies the relation between the curvature of any triple of endpoints and the ratio considered above that two intersecting line segments define.

Three distinct points A, B and C in the plane, not all on a line, lie on a unique circle. Recall that the radius of this circle is

$$R(A,B,C) = \frac{c}{2\sin\gamma}, \tag{1.2}$$

where c is the length of a side of the triangle ABC and γ is the opposite angle. The reciprocal of R is called the *(Menger) curvature* of these three points and is denoted by $K(A,B,C)$.

Now consider two intersecting line segments as in Fig. 2.

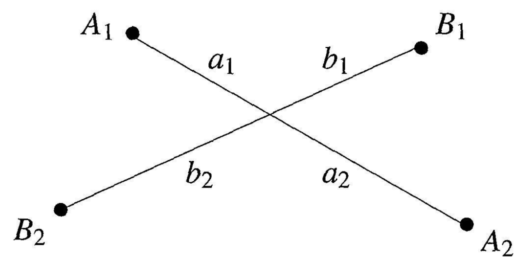

FIGURE 2
Intersecting line segments

PROPOSITION 1.1. *In the above notation, the following equality holds:*

$$\frac{a_1a_2}{b_1b_2} = \frac{K(A_1,B_1,B_2)K(A_2,B_1,B_2)}{K(B_1,A_1,A_2)K(B_2,A_1,A_2)}.$$

Proof. Let α_i be the angle between the line segments A_iB_j and B_1B_2, and let β_i be the angle between B_iA_j and A_1A_2, for $\{i,j\} = \{1,2\}$. By the sine law we have

$$\frac{a_1a_2}{b_1b_2} = \frac{\sin\alpha_1 \sin\alpha_2}{\sin\beta_2 \sin\beta_1} = \frac{2\sin\alpha_1}{|A_1B_1|}\frac{|A_2B_2|}{2\sin\beta_2}\frac{2\sin\alpha_2}{|A_2B_2|}\frac{|A_1B_1|}{2\sin\beta_1}$$
$$= \frac{K(A_1,B_1,B_2)K(A_2,B_1,B_2)}{K(B_1,A_1,A_2)K(B_2,A_1,A_2)}. \qquad \square$$

COROLLARY 1.2. *Let D be a bounded convex domain in $\mathbf{R}^n$. Assume that there is a constant $C > 0$ such that*

$$\frac{K(x,y,z)}{K(x',y',z')} \leq C$$

for any two triples of distinct points in ∂D all lying in the same 2-dimensional plane. Then D satisfies the intersecting chords property.

Proof. Any two intersecting chords define a plane and by Proposition 1.1 we have

$$\frac{a_1a_2}{b_1b_2} = \frac{K_{\alpha 1}K_{\alpha 2}}{K_{\beta 1}K_{\beta 2}} \leq C^2. \qquad \square$$

REMARK 1.3. In view of this subsection it is clear that ICP implies restrictions on the curvature of the boundary, e.g. there cannot be any points of zero curvature. We were however not able to establish the converse of Corollary 1.2.

1.2 CHORDS LARGER THAN δ

The following proposition provides a different approach to the result in [Be97] mentioned above.

PROPOSITION 1.4. *Let D be a bounded convex domain in $\mathbf{R}^n$. Let δ be such that the length of any line segment contained in ∂D is bounded from above by some $\delta' < \delta$. Then there is a constant $C = C(D,\delta) > 0$ such that*

$$C(D,\delta) \leq K(x,y,z) \leq \frac{2}{\delta}, \tag{1.3}$$

whenever $x,y,z \in \partial D$ and $xy \geq \delta$.

Proof. The angle $\alpha(x,y,v) := \angle_y(xy,v)$ is continuous in $x,y \in \mathbf{R}^n$ and $v \in UT_y(\partial D)$, the unit tangent cone at y. The tangent cone at a boundary point y is the union of all hyperplanes containing y but which are disjoint from D. If $[x,y]$ does not lie in ∂D, then $0 < \alpha(x,y,v) < \pi$. The set

$$S = \{(x,y,v) \in \partial D \times \partial D \times UT_y(\partial D) \, : \, xy \geq \delta\}$$