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4. THE TRIPLE RATIO ON  $S$ 

We return to the notation introduced in Sections 1 and 2.

For  $z_1, z_2, z_3 \in \text{Mat}(p \times q, \mathbf{C})$  define, whenever it makes sense, the element  $T(z_1, z_2, z_3) \in \text{GL}(q, \mathbf{C})$  by the following formula

$$(24) \quad \begin{aligned} T(z_1, z_2, z_3) &= k(z_1, z_2) k(z_3, z_2)^{-1} k(z_3, z_1) \\ &= (\mathbf{1}_q - z_2^* z_1)^{-1} (\mathbf{1}_q - z_2^* z_3) (\mathbf{1}_q - z_1^* z_3)^{-1}. \end{aligned}$$

It satisfies the following transformation law

$$(25) \quad T(g(z_1), g(z_2), g(z_3)) = j(g, z_1) T(z_1, z_2, z_3) j(g, z_1)^*$$

for  $g \in G$ . In particular, we see that  $T(\sigma_1, \sigma_2, \sigma_3)$  is well defined on  $S_{\top}^3$  and that the  $\text{GL}(q, \mathbf{C})$ -orbit of  $T(\sigma_1, \sigma_2, \sigma_3)$  is constant along any  $G$ -orbit in  $S_{\top}^3$ .

LEMMA 4.1. Let  $\sigma = \begin{pmatrix} \sigma_p \\ \sigma' \end{pmatrix} \in S$ , tranverse to  $ie$  and  $-ie$ . Then

$$(26) \quad T(ie, -ie, \sigma) = \frac{1}{2i} (i\mathbf{1}_q + \sigma_q) (\mathbf{1}_q + i\sigma_q)^{-1}.$$

*Proof.* This is an easy computation.

PROPOSITION 4.2. Let  $(\sigma_1, \sigma_2, \sigma_3) \in S_{\top}^3$ . Then

$$2i T(\sigma_1, \sigma_2, \sigma_3) \in \tilde{T}_q^{(r)}.$$

*Proof.* Let us first assume  $\sigma_1 = ie, \sigma_2 = -ie, \sigma_3 = \sigma$ . Except for the factor  $\frac{1}{2i}$ , a comparison with (9) shows that  $T(ie, -ie, \sigma)$  is the first term of the Cayley transform of  $\sigma$ . More precisely, let  $c(\sigma) = \xi = \begin{pmatrix} \xi_q \\ \xi' \end{pmatrix}$ . Then we may rewrite (26) as

$$T(ie, -ie, \sigma) = \frac{1}{2i} \xi_q.$$

Now  $\xi$  belongs to  ${}^cS$ , and hence  $\frac{1}{2i}(\xi_q - \xi_q^*) = \xi'^* \xi'$ . But  $\text{rank}(\xi') \leq r$ , so  $\text{rank}(\xi'^* \xi') \leq r$  and hence  $\xi_q$  belongs to  $\tilde{T}_q^{(r)}$ . Now the transformation law (25) for the triple ratio implies that for any  $(\sigma_1, \sigma_2, \sigma_3) \in S_{\top}^3$ ,  $2i T(\sigma_1, \sigma_2, \sigma_3)$  belongs to  $\tilde{T}_q^{(r)}$ .  $\square$

**THEOREM 4.3.** *Let  $(\sigma_1, \sigma_2, \sigma_3)$  and  $(\tau_1, \tau_2, \tau_3)$  belong to  $S^3_{\top}$ . They belong to the same  $G$ -orbit if and only if  $T(\sigma_1, \sigma_2, \sigma_3)$  and  $T(\tau_1, \tau_2, \tau_3)$  belong to the same  $GL(q, \mathbf{C})$ -orbit.*

*Proof.* One way is obvious from the transformation law (25) for the triple ratio. For the converse, we assume (as we may) that  $\sigma_1 = \tau_1 = ie$  and  $\sigma_2 = \tau_2 = -ie$ , and set for simplicity  $\sigma = \sigma_3$  and  $\tau = \tau_3$ . Then the assumption implies that  $(i\mathbf{1}_q + \sigma_q)(\mathbf{1}_q - i\sigma_q)^{-1}$  and  $(i\mathbf{1}_q + \tau_q)(\mathbf{1}_q - i\tau_q)^{-1}$  are in the same  $GL(q, \mathbf{C})$ -orbit. By Lemma 2.3,  $c(\sigma)$  and  $c(\tau)$  are in the same  ${}^cL$ -orbit. So  $\sigma$  and  $\tau$  are in the same  $L$ -orbit.  $\square$

Now to give a description of the invariant in terms of Theorem 3.13, we need to define the analog of the function  $\arg \det$ . For  $z_1 \in D$  and  $z_2 \in \bar{D}$ , the function  $k(z_1, z_2) = (\mathbf{1}_q - z_2^* z_1)^{-1}$  is well defined and belongs to  $GL(q, \mathbf{C})$ . So we can extend the definition of  $T$  to the set

$$\tilde{D}_{\top} = \{(z_1, z_2, z_3) \mid z_i \in D \cup S, 1 \leq i \leq 3, z_1 \top' z_2, z_2 \top' z_3, z_3 \top' z_1\},$$

where by definition  $z \top' w$  is satisfied if  $z$  or  $w$  belongs to  $D$ , and reduces to the condition  $z \top w$  if both  $z$  and  $w$  belong to  $S$ . As  $\tilde{D}_{\top}$  is stable by  $(z_1, z_2, z_3) \mapsto (tz_1, tz_2, tz_3)$  for  $0 \leq t \leq 1$ , this is a simply connected set. For  $z_1 \in D$ ,  $\det T(z_1, z_1, z_1)$  is a positive real number. So there is a well defined continuous determination of the argument of  $\det(T(z_1, z_2, z_3))$  on  $\tilde{D}_{\top}$  such that it takes the value 0 whenever  $z_1 = z_2 = z_3 \in D$ . Denote this determination by  $\arg \det T(z_1, z_2, z_3)$ . It is clearly invariant under the  $G$ -action, and so it defines an invariant for the  $G$ -orbits.

On the other hand, let

$$S(z_1, z_2, z_3) = T(z_1, z_2, z_3)^{*^{-1}} T(z_1, z_2, z_3)$$

be the angular matrix associated to  $T(z_1, z_2, z_3)$ .

**THEOREM 4.4.** *Let  $(\sigma_1, \sigma_2, \sigma_3)$  and  $(\tau_1, \tau_2, \tau_3)$  belong to  $S^3_{\top}$ . They belong to the same  $G$ -orbit if and only if  $S(\sigma_1, \sigma_2, \sigma_3)$  and  $S(\tau_1, \tau_2, \tau_3)$  are conjugate under  $GL(q, \mathbf{C})$  and  $\arg \det T(\sigma_1, \sigma_2, \sigma_3) = \arg \det T(\tau_1, \tau_2, \tau_3)$ .*

*Proof.* This is a direct consequence of Theorem 4.3 and Theorem 3.13.

**REMARK 1.** Let us consider the case where  $q = 1$ . The Stiefel manifold is the unit sphere  $S^{2p-1}$  in  $\mathbf{C}^p$ . The transversality condition  $\sigma \top \tau$  just means  $\sigma \neq \tau$ , as is easily seen from the Cauchy-Schwarz inequality. The triple ratio

is the complex number

$$T(\sigma_1, \sigma_2, \sigma_3) = (1 - \sigma_2^* \sigma_1)^{-1} (1 - \sigma_2^* \sigma_3) (1 - \sigma_1^* \sigma_3)^{-1}.$$

The group  $GL(q, \mathbf{C}) \simeq \mathbf{C}^*$  acts on the upper halfplane by  $(\lambda, z) \mapsto |\lambda|^2 z$  and so the orbits are described by the argument of the complex number  $z$ . So the characteristic invariant in this case is just

$$\arg \left( (1 - \sigma_2^* \sigma_1)^{-1} (1 - \sigma_2^* \sigma_3) (1 - \sigma_1^* \sigma_3)^{-1} \right).$$

It is equivalent to the invariant  $\theta$  considered in [KR]. This invariant, almost in our terms, was known to E. Cartan (see [Ca]).

REMARK 2. Let us consider the case where  $p = q$ . Then the Stiefel manifold is  $U(q)$ , and the content of Proposition 4.2 is that for  $(\sigma_1, \sigma_2, \sigma_3) \in S_{\top}^3$

$$T(\sigma_1, \sigma_2, \sigma_3) = (1 - \sigma_2^* \sigma_1)^{-1} (1 - \sigma_2^* \sigma_3) (1 - \sigma_1^* \sigma_3)^{-1}$$

is an invertible skew-Hermitian matrix. The orbits of  $GL(q, \mathbf{C})$  in its action on nondegenerate Hermitian forms are characterized by the signature. So the characteristic invariant as described in Theorem 4.3 in this case reduces to  $\text{sgn } iT(\sigma_1, \sigma_2, \sigma_3)$ . As concerns Theorem 4.4, notice that the invariant  $S$  is trivial (equal to  $-\mathbf{1}_q$ ), so one is only concerned with the invariant  $\arg \det T$ . The bounded domain  $D$  is of tube type and the description of the invariant through the function  $\arg \det$  coincides with the approach of this problem in [CØ], where the invariant was introduced under the name of *generalized Maslov index*.

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