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4. THE TRIPLE RATIO ON S

We return to the notation introduced in Sections 1 and 2.

For $z_1, z_2, z_3 \in \mathrm{Mat}(p \times q, \mathbf{C})$ define, whenever it makes sense, the element $T(z_1, z_2, z_3) \in \mathrm{GL}(q, \mathbf{C})$ by the following formula

$$(24) \qquad \begin{aligned} T(z_1, z_2, z_3) &= k(z_1, z_2)\, k(z_3, z_2)^{-1} k(z_3, z_1) \\ &= (\mathbf{1}_q - z_2^* z_1)^{-1} (\mathbf{1}_q - z_2^* z_3)(\mathbf{1}_q - z_1^* z_3)^{-1} . \end{aligned}$$

It satisfies the following transformation law

$$(25) \qquad T(g(z_1), g(z_2), g(z_3)) = j(g, z_1)\, T(z_1, z_2, z_3) j(g, z_1)^*$$

for $g \in G$. In particular, we see that $T(\sigma_1, \sigma_2, \sigma_3)$ is well defined on $S^3_{\top}$ and that the $\mathrm{GL}(q, \mathbf{C})$-orbit of $T(\sigma_1, \sigma_2, \sigma_3)$ is constant along any G-orbit in $S^3_{\top}$.

LEMMA 4.1. *Let* $\sigma = \begin{pmatrix} \sigma_p \\ \sigma' \end{pmatrix} \in S$, *tranverse to* ie *and* $-ie$. *Then*

$$(26) \qquad T(ie, -ie, \sigma) = \frac{1}{2i} (i\mathbf{1}_q + \sigma_q)(\mathbf{1}_q + i\sigma_q)^{-1} .$$

Proof. This is an easy computation.

PROPOSITION 4.2. *Let* $(\sigma_1, \sigma_2, \sigma_3) \in S^3_{\top}$. *Then*

$$2i\ T(\sigma_1, \sigma_2, \sigma_3) \in \widetilde{T}_q^{(r)} .$$

Proof. Let us first assume $\sigma_1 = ie, \sigma_2 = -ie, \sigma_3 = \sigma$. Except for the factor $\frac{1}{2i}$, a comparison with (9) shows that $T(ie, -ie, \sigma)$ is the first term of the Cayley transform of σ. More precisely, let $c(\sigma) = \xi = \begin{pmatrix} \xi_q \\ \xi' \end{pmatrix}$. Then we may rewrite (26) as

$$T(ie, -ie, \sigma) = \frac{1}{2i} \xi_q .$$

Now ξ belongs to ${}^cS$, and hence $\frac{1}{2i}(\xi_q - \xi_q^*) = \xi'^* \xi'$. But $\mathrm{rank}(\xi') \leq r$, so $\mathrm{rank}(\xi'^* \xi') \leq r$ and hence ξ_q belongs to $\widetilde{T}_q^{(r)}$. Now the transformation law (25) for the triple ratio implies that for *any* $(\sigma_1, \sigma_2, \sigma_3) \in S^3_{\top}$, $2i\, T(\sigma_1, \sigma_2, \sigma_3)$ belongs to $\widetilde{T}_q^{(r)}$. □

THEOREM 4.3. *Let* $(\sigma_1, \sigma_2, \sigma_3)$ *and* (τ_1, τ_2, τ_3) *belong to* $S^3_\top$. *They belong to the same* G*-orbit if and only if* $T(\sigma_1, \sigma_2, \sigma_3)$ *and* $T(\tau_1, \tau_2, \tau_3)$ *belong to the same* $\mathrm{GL}(q, \mathbf{C})$*-orbit.*

Proof. One way is obvious from the transformation law (25) for the triple ratio. For the converse, we assume (as we may) that $\sigma_1 = \tau_1 = ie$ and $\sigma_2 = \tau_2 = -ie$, and set for simplicity $\sigma = \sigma_3$ and $\tau = \tau_3$. Then the assumption implies that $(i\mathbf{1}_q + \sigma_q)(\mathbf{1}_q - i\sigma_q)^{-1}$ and $(i\mathbf{1}_q + \tau_q)(\mathbf{1}_q - i\tau_q)^{-1}$ are in the same $\mathrm{GL}(q, \mathbf{C})$-orbit. By Lemma 2.3, $c(\sigma)$ and $c(\tau)$ are in the same ${}^cL$-orbit. So σ and τ are in the same L-orbit. □

Now to give a description of the invariant in terms of Theorem 3.13, we need to define the analog of the function arg det. For $z_1 \in D$ and $z_2 \in \overline{D}$, the function $k(z_1, z_2) = (\mathbf{1}_q - z_2^* z_1)^{-1}$ is well defined and belongs to $\mathrm{GL}(q, \mathbf{C})$. So we can extend the definition of T to the set

$$\widetilde{D}_\top = \{(z_1, z_2, z_3) \mid z_i \in D \cup S,\ 1 \le i \le 3,\ z_1 \top' z_2,\ z_2 \top' z_3,\ z_3 \top' z_1\},$$

where by definition $z \top' w$ is satisfied if z or w belongs to D, and reduces to the condition $z \top w$ if both z and w belong to S. As $\widetilde{D}_\top$ is stable by $(z_1, z_2, z_3) \mapsto (tz_1, tz_2, tz_3)$ for $0 \le t \le 1$, this is a simply connected set. For $z_1 \in D$, $\det T(z_1, z_1, z_1)$ is a positive real number. So there is a well defined continuous determination of the argument of $\det(T(z_1, z_2, z_3))$ on $\widetilde{D}_\top$ such that it takes the value 0 whenever $z_1 = z_2 = z_3 \in D$. Denote this determination by $\arg\det T(z_1, z_2, z_3)$. It is clearly invariant under the G-action, and so it defines an invariant for the G-orbits.

On the other hand, let

$$S(z_1, z_2, z_3) = T(z_1, z_2, z_3)^{*^{-1}} T(z_1, z_2, z_3)$$

be the angular matrix associated to $T(z_1, z_2, z_3)$.

THEOREM 4.4. *Let* $(\sigma_1, \sigma_2, \sigma_3)$ *and* (τ_1, τ_2, τ_3) *belong to* $S^3_\top$. *They belong to the same* G*-orbit if and only if* $S(\sigma_1, \sigma_2, \sigma_3)$ *and* $S(\tau_1, \tau_2, \tau_3)$ *are conjugate under* $\mathrm{GL}(q, \mathbf{C})$ *and* $\arg\det T(\sigma_1, \sigma_2, \sigma_3) = \arg\det T(\tau_1, \tau_2, \tau_3)$.

Proof. This is a direct consequence of Theorem 4.3 and Theorem 3.13.

REMARK 1. Let us consider the case where $q = 1$. The Stiefel manifold is the unit sphere $\mathbf{S}^{2p-1}$ in $\mathbf{C}^p$. The transversality condition $\sigma \top \tau$ just means $\sigma \neq \tau$, as is easily seen from the Cauchy-Schwarz inequality. The triple ratio is the complex number

$$T(\sigma_1,\sigma_2,\sigma_3) = (1-\sigma_2^*\sigma_1)^{-1}(1-\sigma_2^*\sigma_3)(1-\sigma_1^*\sigma_3)^{-1}\,.$$

The group $\mathrm{GL}(q,\mathbf{C}) \simeq \mathbf{C}^*$ acts on the upper halfplane by $(\lambda, z) \mapsto |\lambda|^2 z$ and so the orbits are described by the argument of the complex number z. So the characteristic invariant in this case is just

$$\arg\left((1-\sigma_2^*\sigma_1)^{-1}(1-\sigma_2^*\sigma_3)(1-\sigma_1^*\sigma_3)^{-1}\right).$$

It is equivalent to the invariant θ considered in [KR]. This invariant, almost in our terms, was known to E. Cartan (see [Ca]).

REMARK 2. Let us consider the case where $p = q$. Then the Stiefel manifold is $\mathrm{U}(q)$, and the content of Proposition 4.2 is that for $(\sigma_1,\sigma_2,\sigma_3) \in S^3_{\mathrm{T}}$

$$T(\sigma_1,\sigma_2,\sigma_3) = (1-\sigma_2^*\sigma_1)^{-1}(1-\sigma_2^*\sigma_3)(1-\sigma_1^*\sigma_3)^{-1}$$

is an invertible skew-Hermitian matrix. The orbits of $\mathrm{GL}(q,\mathbf{C})$ in its action on nondegenerate Hermitian forms are characterized by the signature. So the characteristic invariant as described in Theorem 4.3 in this case reduces to $\operatorname{sgn} iT(\sigma_1,\sigma_2,\sigma_3)$. As concerns Theorem 4.4, notice that the invariant S is trivial (equal to $-\mathbf{1}_q$), so one is only concerned with the invariant $\arg\det T$. The bounded domain D is of tube type and the description of the invariant through the function $\arg\det$ coincides with the approach of this problem in [CØ], where the invariant was introduced under the name of *generalized Maslov index.*

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