

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 48 (2002)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: TRIPLE RATIO ON THE UNITARY STIEFEL MANIFOLD
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Kapitel: 4. The triple ratio on S
DOI: <https://doi.org/10.5169/seals-66067>

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4. THE TRIPLE RATIO ON S

We return to the notation introduced in Sections 1 and 2.

For $z_1, z_2, z_3 \in \text{Mat}(p \times q, \mathbf{C})$ define, whenever it makes sense, the element $T(z_1, z_2, z_3) \in \text{GL}(q, \mathbf{C})$ by the following formula

$$(24) \quad \begin{aligned} T(z_1, z_2, z_3) &= k(z_1, z_2) k(z_3, z_2)^{-1} k(z_3, z_1) \\ &= (\mathbf{1}_q - z_2^* z_1)^{-1} (\mathbf{1}_q - z_2^* z_3) (\mathbf{1}_q - z_1^* z_3)^{-1}. \end{aligned}$$

It satisfies the following transformation law

$$(25) \quad T(g(z_1), g(z_2), g(z_3)) = j(g, z_1) T(z_1, z_2, z_3) j(g, z_1)^*$$

for $g \in G$. In particular, we see that $T(\sigma_1, \sigma_2, \sigma_3)$ is well defined on S_{\top}^3 and that the $\text{GL}(q, \mathbf{C})$ -orbit of $T(\sigma_1, \sigma_2, \sigma_3)$ is constant along any G -orbit in S_{\top}^3 .

LEMMA 4.1. *Let $\sigma = \begin{pmatrix} \sigma_p \\ \sigma' \end{pmatrix} \in S$, transverse to ie and $-ie$. Then*

$$(26) \quad T(ie, -ie, \sigma) = \frac{1}{2i} (i\mathbf{1}_q + \sigma_q) (\mathbf{1}_q + i\sigma_q)^{-1}.$$

Proof. This is an easy computation.

PROPOSITION 4.2. *Let $(\sigma_1, \sigma_2, \sigma_3) \in S_{\top}^3$. Then*

$$2i T(\sigma_1, \sigma_2, \sigma_3) \in \widetilde{T}_q^{(r)}.$$

Proof. Let us first assume $\sigma_1 = ie, \sigma_2 = -ie, \sigma_3 = \sigma$. Except for the factor $\frac{1}{2i}$, a comparison with (9) shows that $T(ie, -ie, \sigma)$ is the first term of the Cayley transform of σ . More precisely, let $c(\sigma) = \xi = \begin{pmatrix} \xi_q \\ \xi' \end{pmatrix}$. Then we may rewrite (26) as

$$T(ie, -ie, \sigma) = \frac{1}{2i} \xi_q.$$

Now ξ belongs to ${}^c S$, and hence $\frac{1}{2i} (\xi_q - \xi_q^*) = \xi'^* \xi'$. But $\text{rank}(\xi') \leq r$, so $\text{rank}(\xi'^* \xi') \leq r$ and hence ξ_q belongs to $\widetilde{T}_q^{(r)}$. Now the transformation law (25) for the triple ratio implies that for any $(\sigma_1, \sigma_2, \sigma_3) \in S_{\top}^3$, $2i T(\sigma_1, \sigma_2, \sigma_3)$ belongs to $\widetilde{T}_q^{(r)}$. \square

THEOREM 4.3. *Let $(\sigma_1, \sigma_2, \sigma_3)$ and (τ_1, τ_2, τ_3) belong to S_{\top}^3 . They belong to the same G -orbit if and only if $T(\sigma_1, \sigma_2, \sigma_3)$ and $T(\tau_1, \tau_2, \tau_3)$ belong to the same $\mathrm{GL}(q, \mathbf{C})$ -orbit.*

Proof. One way is obvious from the transformation law (25) for the triple ratio. For the converse, we assume (as we may) that $\sigma_1 = \tau_1 = ie$ and $\sigma_2 = \tau_2 = -ie$, and set for simplicity $\sigma = \sigma_3$ and $\tau = \tau_3$. Then the assumption implies that $(i\mathbf{1}_q + \sigma_q)(\mathbf{1}_q - i\sigma_q)^{-1}$ and $(i\mathbf{1}_q + \tau_q)(\mathbf{1}_q - i\tau_q)^{-1}$ are in the same $\mathrm{GL}(q, \mathbf{C})$ -orbit. By Lemma 2.3, $c(\sigma)$ and $c(\tau)$ are in the same ${}^c L$ -orbit. So σ and τ are in the same L -orbit. \square

Now to give a description of the invariant in terms of Theorem 3.13, we need to define the analog of the function $\arg \det$. For $z_1 \in D$ and $z_2 \in \bar{D}$, the function $k(z_1, z_2) = (\mathbf{1}_q - z_2^* z_1)^{-1}$ is well defined and belongs to $\mathrm{GL}(q, \mathbf{C})$. So we can extend the definition of T to the set

$$\tilde{D}_{\top} = \{(z_1, z_2, z_3) \mid z_i \in D \cup S, 1 \leq i \leq 3, z_1 \top' z_2, z_2 \top' z_3, z_3 \top' z_1\},$$

where by definition $z \top' w$ is satisfied if z or w belongs to D , and reduces to the condition $z \top w$ if both z and w belong to S . As \tilde{D}_{\top} is stable by $(z_1, z_2, z_3) \mapsto (tz_1, tz_2, tz_3)$ for $0 \leq t \leq 1$, this is a simply connected set. For $z_1 \in D$, $\det T(z_1, z_1, z_1)$ is a positive real number. So there is a well defined continuous determination of the argument of $\det(T(z_1, z_2, z_3))$ on \tilde{D}_{\top} such that it takes the value 0 whenever $z_1 = z_2 = z_3 \in D$. Denote this determination by $\arg \det T(z_1, z_2, z_3)$. It is clearly invariant under the G -action, and so it defines an invariant for the G -orbits.

On the other hand, let

$$S(z_1, z_2, z_3) = T(z_1, z_2, z_3)^{*^{-1}} T(z_1, z_2, z_3)$$

be the angular matrix associated to $T(z_1, z_2, z_3)$.

THEOREM 4.4. *Let $(\sigma_1, \sigma_2, \sigma_3)$ and (τ_1, τ_2, τ_3) belong to S_{\top}^3 . They belong to the same G -orbit if and only if $S(\sigma_1, \sigma_2, \sigma_3)$ and $S(\tau_1, \tau_2, \tau_3)$ are conjugate under $\mathrm{GL}(q, \mathbf{C})$ and $\arg \det T(\sigma_1, \sigma_2, \sigma_3) = \arg \det T(\tau_1, \tau_2, \tau_3)$.*

Proof. This is a direct consequence of Theorem 4.3 and Theorem 3.13.

REMARK 1. Let us consider the case where $q = 1$. The Stiefel manifold is the unit sphere S^{2p-1} in \mathbf{C}^p . The transversality condition $\sigma \top \tau$ just means $\sigma \neq \tau$, as is easily seen from the Cauchy-Schwarz inequality. The triple ratio

is the complex number

$$T(\sigma_1, \sigma_2, \sigma_3) = (1 - \sigma_2^* \sigma_1)^{-1} (1 - \sigma_2^* \sigma_3) (1 - \sigma_1^* \sigma_3)^{-1}.$$

The group $\mathrm{GL}(q, \mathbf{C}) \simeq \mathbf{C}^*$ acts on the upper halfplane by $(\lambda, z) \mapsto |\lambda|^2 z$ and so the orbits are described by the argument of the complex number z . So the characteristic invariant in this case is just

$$\arg((1 - \sigma_2^* \sigma_1)^{-1} (1 - \sigma_2^* \sigma_3) (1 - \sigma_1^* \sigma_3)^{-1}).$$

It is equivalent to the invariant θ considered in [KR]. This invariant, almost in our terms, was known to E. Cartan (see [Ca]).

REMARK 2. Let us consider the case where $p = q$. Then the Stiefel manifold is $\mathrm{U}(q)$, and the content of Proposition 4.2 is that for $(\sigma_1, \sigma_2, \sigma_3) \in S_{\top}^3$

$$T(\sigma_1, \sigma_2, \sigma_3) = (1 - \sigma_2^* \sigma_1)^{-1} (1 - \sigma_2^* \sigma_3) (1 - \sigma_1^* \sigma_3)^{-1}$$

is an invertible skew-Hermitian matrix. The orbits of $\mathrm{GL}(q, \mathbf{C})$ in its action on nondegenerate Hermitian forms are characterized by the signature. So the characteristic invariant as described in Theorem 4.3 in this case reduces to $\mathrm{sgn} iT(\sigma_1, \sigma_2, \sigma_3)$. As concerns Theorem 4.4, notice that the invariant S is trivial (equal to $-\mathbf{1}_q$), so one is only concerned with the invariant $\arg \det T$. The bounded domain D is of tube type and the description of the invariant through the function $\arg \det$ coincides with the approach of this problem in [CØ], where the invariant was introduced under the name of *generalized Maslov index*.

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