

2. Action of G on $S \times S$ and $S \times S \times S$

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **48 (2002)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **21.09.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

2. ACTION OF G ON $S \times S$ AND $S \times S \times S$

We now study the action of G on pairs of points of S . The main notion to be introduced is *transversality*, a notion that could be defined for any bounded symmetric domain. We give several equivalent definitions for our case.

PROPOSITION 2.1. *Let σ and ξ be two elements of S . Then the following are equivalent:*

- (i) $\det(\mathbf{1}_q - \xi^* \sigma) \neq 0$;
- (ii) $\xi - \sigma$ injective;
- (iii) $\det(\mathbf{1}_p - \xi \sigma^*) \neq 0$.

If one of these equivalent conditions is satisfied, then σ and ξ are said to be transverse.

Proof. Assume (i). As $\mathbf{1}_q = \xi^* \xi$, this condition amounts to $\det(\xi^*(\xi - \sigma)) \neq 0$, which in particular shows that $\xi - \sigma$ is injective. Conversely, assume $\xi - \sigma$ is injective and let $v \in \mathbf{C}^q$ be such that $v = \xi^* \sigma v$. Now

$$\|v\| = \|\xi^* \sigma v\| \leq \|\sigma v\| \leq \|v\|,$$

and hence $\|\xi^* \sigma v\| = \|\sigma v\|$, which is possible only if $\sigma v \in \text{Im } \xi$. So there exists $w \in \mathbf{C}^q$, such that $\sigma v = \xi w$. But taking the image of both sides by ξ^* yields $v = w$, and hence $\sigma v = \xi v$, so that $v = 0$. So $\mathbf{1}_q - \xi^* \sigma$ is injective and hence (ii) \implies (i). Under the same assumption (ii), let us prove that $\xi \sigma^*$ cannot have 1 as an eigenvalue. Suppose $v \in \mathbf{C}^p$ is such that $\xi \sigma^* v = v$. As ξ is a partial isometry, this forces $\|\sigma^* v\| = \|v\|$, and hence v belongs to the image of the map σ , so there exists $w \in \mathbf{C}^q$ such that $v = \sigma w$. But then we also have $v = \xi \sigma^* \sigma w = \xi w$ and hence $(\sigma - \xi)w = 0$ which forces $w = 0$. Hence (iii) follows from (ii). Finally assume (iii). Then as σ is injective, $(\mathbf{1}_p - \xi \sigma^*) \circ \sigma = \sigma - \xi$ is also injective. Hence (iii) \implies (ii). \square

We will use the notation $\sigma \top \xi$ to denote transversality. It is a symmetric condition. It is invariant under the action of G , as can easily be concluded from (6). For $\sigma \in S$, let

$$S_{\top}^{\sigma} = \{\xi \mid \sigma \top \xi\}.$$

Observe that the set S_{\top}^{ie} is exactly the subset in S where the Cayley transform is defined.

Let

$$(14) \quad S_{\top}^2 = \{(\sigma, \xi) \in S \times S \mid \sigma \top \xi\}.$$

As base point in S_{\top}^2 we choose $(ie, -ie)$. Observe that $c(-ie) = 0$.

THEOREM 2.2. *The group G acts transitively on S_{\top}^2 .*

Proof. Let $(\sigma, \xi) \in S_{\top}^2$ and let us show that there exists an element of G which maps (σ, ξ) to $(ie, -ie)$. As G is transitive on S , we may assume that $\sigma = ie$. Then the transversality condition shows that ξ belongs to the domain of the Cayley transform. The element $c(\xi)$ belongs to cS , and we have already noticed that cB is transitive on cS . Hence $c(\xi)$ can be mapped to $0 = c(-ie)$. Taking the image under the inverse Cayley transform gives the result. \square

Denote by L the stabilizer of the base point $(ie, -ie)$ in B . Under a Cayley transform, the group ${}^cL = c \circ L \circ c^{-1}$ is the stabilizer in cB of the element 0. Hence it is the subgroup of linear transformations given by

$$\begin{aligned} w_q &\longmapsto h^* w_q h \\ w' &\longmapsto u w h \end{aligned}$$

where $h \in \text{GL}(q, \mathbf{C})$, $u \in \text{U}(p - q)$ and $\det h = (\det u)^{-1}$.

LEMMA 2.3. *Let $\begin{pmatrix} w_q \\ w' \end{pmatrix}, \begin{pmatrix} v_q \\ v' \end{pmatrix} \in {}^cS$. Then they belong to the same orbit under the action of cL if and only if w_q and v_q belong to the same orbit under the action of $\text{GL}(q, \mathbf{C})$.*

Proof. One implication being trivial, we only have to prove the other one. So assume there exists $h \in \text{GL}(q, \mathbf{C})$ such that $v_q = h^* w_q h$. Let μ be a complex number such that $\mu^{p-q} = \det h$ and let $u = \mu^{-1} \mathbf{1}_{p-q}$. Clearly $(\det u)^{-1} = \det h$. Using the action of (h, u) we may assume that $v_q = w_q$. Let $s_q = \frac{1}{2i}(w_q - w_q^*)$. This is an Hermitian matrix and as w_q and v_q belong to cS , we get

$$s_q = w'^* w' = v'^* v'.$$

Looking to the columns of w' (or v'), we may think of w' as a family of q vectors in \mathbf{C}^{p-q} . Then the matrix s_q is the Gram matrix of these vectors. But two sets of vectors in \mathbf{C}^{p-q} are conjugate under the action of the unitary group $\text{U}(p - q)$ if and only if they have the same Gram matrix. Hence there exists $u \in \text{U}(p - q)$ such that $v' = u w'$. Let λ be a complex number such that $\lambda^q = \det u$. Then using the action of $(\lambda^{-1} \mathbf{1}_q, u)$, we get the result. \square

Let us denote by H_q the real vector space of $q \times q$ Hermitian matrices, and let Ω_q be the subset of all positive-definite matrices. For any integer r such that $0 \leq r \leq q$ let $\Omega_q^{(r)}$ be the set of all positive semi-definite $q \times q$ Hermitian matrices of rank less than r . For $r < q$, the set $\Omega_q^{(r)}$ is contained in the boundary of Ω_q , whereas for $r = q$, $\Omega_q^{(q)} = \overline{\Omega}_q$.

Let

$$T_q^{(r)} = \{x + iy \mid x \in H_q, y \in \Omega_q^{(r)}\}.$$

The group $GL(q, \mathbf{C})$ acts on $T_q^{(r)}$ by the action $(h, w) \mapsto hwh^*$.

Finally let

$$\tilde{T}_q^{(r)} = \{z \in T_q^{(r)} \mid z \text{ invertible}\}.$$

Clearly the action of $GL(q, \mathbf{C})$ preserves $\tilde{T}_q^{(r)}$.

Let $\begin{pmatrix} w_q \\ w' \end{pmatrix}$ be in cS . Then $w_q = x_q + iw'^*w'$, with $x_q \in H_q$. Let

$$r = \inf(q, p - q).$$

The rank of the matrix w'^*w' is at most r . Hence w_q belongs to $T_q^{(r)}$. Conversely, it is easily seen that any positive semi-definite Hermitian matrix of rank at most r can be written as w'^*w' for some $w' \in \text{Mat}((p - q) \times q, \mathbf{C})$.

Let

$$(15) \quad S_{\top}^3 = \{(\sigma_1, \sigma_2, \sigma_3) \in S \times S \times S \mid \sigma_1 \top \sigma_2, \sigma_2 \top \sigma_3, \sigma_3 \top \sigma_1\}.$$

THEOREM 2.4. *The G -orbits in S_{\top}^3 are in one-to-one correspondance with the orbits of $GL(q, \mathbf{C})$ in $\tilde{T}_q^{(r)}$.*

Proof. From Theorem 2.2 we already know that any orbit contains an element of the form $(ie, -ie, \sigma)$ with $\sigma \in S$. Now use the Cayley transform. The element $w = c(\sigma)$ is in cS , and the transversality condition is equivalent to the condition $\det(w_q) \neq 0$. In other words, $w_q \in \tilde{T}_q^{(r)}$. The result now follows from Lemma 2.3. \square