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2. ACTION OF G ON $S \times S$ AND $S \times S \times S$

We now study the action of G on pairs of points of S. The main notion to be introduced is *transversality*, a notion that could be defined for any bounded symmetric domain. We give several equivalent definitions for our case.

PROPOSITION 2.1. *Let σ and ξ be two elements of S. Then the following are equivalent:*

(i) $\det(\mathbf{1}_q - \xi^*\sigma) \neq 0$;

(ii) $\xi - \sigma$ *injective;*

(iii) $\det(\mathbf{1}_p - \xi\sigma^*) \neq 0$.

If one of these equivalent conditions is satisfied, then σ and ξ are said to be transverse.

Proof. Assume (i). As $\mathbf{1}_q = \xi^*\xi$, this condition amounts to $\det(\xi^*(\xi-\sigma)) \neq 0$, which in particular shows that $\xi - \sigma$ is injective. Conversely, assume $\xi - \sigma$ is injective and let $v \in \mathbf{C}^q$ be such that $v = \xi^*\sigma v$. Now

$$\|v\| = \|\xi^*\sigma v\| \leq \|\sigma v\| \leq \|v\|,$$

and hence $\|\xi^*\sigma v\| = \|\sigma v\|$, which is possible only if $\sigma v \in \operatorname{Im} \xi$. So there exists $w \in \mathbf{C}^q$, such that $\sigma v = \xi w$. But taking the image of both sides by ξ^* yields $v = w$, and hence $\sigma v = \xi v$, so that $v = 0$. So $\mathbf{1}_q - \xi^*\sigma$ is injective and hence (ii) $\Longrightarrow$ (i). Under the same assumption (ii), let us prove that $\xi\sigma^*$ cannot have 1 as an eigenvalue. Suppose $v \in \mathbf{C}^p$ is such that $\xi\sigma^* v = v$. As ξ is a partial isometry, this forces $\|\sigma^* v\| = \|v\|$, and hence v belongs to the image of the map σ, so there exists $w \in \mathbf{C}^q$ such that $v = \sigma w$. But then we also have $v = \xi\sigma^*\sigma w = \xi w$ and hence $(\sigma - \xi)w = 0$ which forces $w = 0$. Hence (iii) follows from (ii). Finally assume (iii). Then as σ is injective, $(\mathbf{1}_p - \xi\sigma^*) \circ \sigma = \sigma - \xi$ is also injective. Hence (iii) $\Longrightarrow$ (ii). $\square$

We will use the notation $\sigma \top \xi$ to denote transversality. It is a symmetric condition. It is invariant under the action of G, as can easily be concluded from (6). For $\sigma \in S$, let

$$S_{\top}^{\sigma} = \{\xi \mid \sigma \top \xi\}.$$

Observe that the set $S_{\top}^{ie}$ is exactly the subset in S where the Cayley transform is defined.

Let

(14) $$S^2_\top = \{(\sigma,\xi) \in S \times S \mid \sigma \top \xi\}\,.$$

As base point in $S^2_\top$ we choose $(ie, -ie)$. Observe that $c(-ie) = 0$.

THEOREM 2.2. *The group G acts transitively on $S^2_\top$.*

Proof. Let $(\sigma,\xi) \in S^2_\top$ and let us show that there exists an element of G which maps (σ,ξ) to $(ie,-ie)$. As G is transitive on S, we may assume that $\sigma = ie$. Then the transversality condition shows that ξ belongs to the domain of the Cayley transform. The element $c(\xi)$ belongs to ${}^cS$, and we have already noticed that ${}^cB$ is transitive on ${}^cS$. Hence $c(\xi)$ can be mapped to $0 = c(-ie)$. Taking the image under the inverse Cayley transform gives the result. $\square$

Denote by L the stabilizer of the base point $(ie,-ie)$ in B. Under a Cayley transform, the group ${}^cL = c \circ L \circ c^{-1}$ is the stabilizer in ${}^cB$ of the element 0. Hence it is the subgroup of linear transformations given by

$$\begin{aligned} w_q &\longmapsto h^* w_q h \\ w' &\longmapsto u w h \end{aligned}$$

where $h \in \mathrm{GL}(q,\mathbf{C})$, $u \in \mathrm{U}(p-q)$ and $\det h = (\det u)^{-1}$.

LEMMA 2.3. *Let $\begin{pmatrix} w_q \\ w' \end{pmatrix}, \begin{pmatrix} v_q \\ v' \end{pmatrix} \in {}^cS$. Then they belong to the same orbit under the action of ${}^cL$ if and only if w_q and v_q belong to the same orbit under the action of* $\mathrm{GL}(q,\mathbf{C})$.

Proof. One implication being trivial, we only have to prove the other one. So assume there exists $h \in \mathrm{GL}(q,\mathbf{C})$ such that $v_q = h^* w_q h$. Let μ be a complex number such that $\mu^{p-q} = \det h$ and let $u = \mu^{-1}\mathbf{1}_{p-q}$. Clearly $(\det u)^{-1} = \det h$. Using the action of (h,u) we may assume that $v_q = w_q$. Let $s_q = \frac{1}{2i}(w_q - w_q^*)$. This is an Hermitian matrix and as w_q and v_q belong to ${}^cS$, we get

$$s_q = w'^* w' = v'^* v'\,.$$

Looking to the columns of w' (or v'), we may think of w' as a family of q vectors in $\mathbf{C}^{p-q}$. Then the matrix s_q is the Gram matrix of these vectors. But two sets of vectors in $\mathbf{C}^{p-q}$ are conjugate under the action of the unitary group $\mathrm{U}(p-q)$ if and only if they have the same Gram matrix. Hence there exists $u \in \mathrm{U}(p-q)$ such that $v' = uw'$. Let λ be a complex number such that $\lambda^q = \det u$. Then using the action of $(\lambda^{-1}\mathbf{1}_q, u)$, we get the result. $\square$

Let us denote by H_q the real vector space of $q \times q$ Hermitian matrices, and let Ω_q be the subset of all positive-definite matrices. For any integer r such that $0 \leq r \leq q$ let $\Omega_q^{(r)}$ be the set of all positive semi-definite $q \times q$ Hermitian matrices of rank less than r. For $r < q$, the set $\Omega_q^{(r)}$ is contained in the boundary of Ω_q, whereas for $r = q$, $\Omega_q^{(q)} = \overline{\Omega}_q$.

Let

$$T_q^{(r)} = \{x + iy \mid x \in H_q, y \in \Omega_q^{(r)}\}.$$

The group $\mathrm{GL}(q, \mathbf{C})$ acts on $T_q^{(r)}$ by the action $(h, w) \longmapsto hwh^*$.

Finally let

$$\widetilde{T}_q^{(r)} = \{z \in T_q^{(r)} \mid z \text{ invertible}\}.$$

Clearly the action of $\mathrm{GL}(q, \mathbf{C})$ preserves $\widetilde{T}_q^{(r)}$.

Let $\begin{pmatrix} w_q \\ w' \end{pmatrix}$ be in ${}^cS$. Then $w_q = x_q + iw'^*w'$, with $x_q \in H_q$. Let

$$r = \inf(q, p - q).$$

The rank of the matrix w'^*w' is at most r. Hence w_q belongs to $T_q^{(r)}$. Conversely, it is easily seen that any positive semi-definite Hermitian matrix of rank at most r can be written as w'^*w' for some $w' \in \mathrm{Mat}((p-q) \times q, \mathbf{C})$.

Let

$$\text{(15)} \qquad S_\top^3 = \{(\sigma_1, \sigma_2, \sigma_3) \in S \times S \times S \mid \sigma_1 \top \sigma_2, \sigma_2 \top \sigma_3, \sigma_3 \top \sigma_1\}.$$

THEOREM 2.4. *The G-orbits in $S_\top^3$ are in one-to-one correspondance with the orbits of* $\mathrm{GL}(q, \mathbf{C})$ *in* $\widetilde{T}_q^{(r)}$.

Proof. From Theorem 2.2 we already know that any orbit contains an element of the form $(ie, -ie, \sigma)$ with $\sigma \in S$. Now use the Cayley transform. The element $w = c(\sigma)$ is in ${}^cS$, and the transversality condition is equivalent to the condition $\det(w_q) \neq 0$. In other words, $w_q \in \widetilde{T}_q^{(r)}$. The result now follows from Lemma 2.3. $\square$