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is the unit ball in the matrix space  $\text{Mat}(p \times q, \mathbf{C})$ ,  $S$  is the unitary Stiefel manifold  $\mathbf{S}_{p,q}$  and  $G = \mathbf{PSU}(p, q)$ . The invariant we construct for triples is of matrix-valued nature (it is a conjugacy class) and we give two versions of it (see Theorems 4.3 and 4.4). The basic strategy is to approach the Shilov boundary from inside. The (matrix-valued) *automorphy kernel* for the domain  $D$  is used to build a kernel for triples of points inside  $D$  which transforms nicely under the action of  $G$ . It remains to look carefully at the boundary behaviour of the kernel when the points approach the Shilov boundary  $S$ . This is only possible for triples satisfying a generic condition called *transversality* (see Proposition 2.1 for a definition). The *Cayley transform* plays an important role in the proofs. Finally the problem is reduced to a *linear* problem, which is related to the description of some orbits for the action  $(g, X) \mapsto gXg^*$  of  $\text{GL}_q$  on  $\text{Mat}(q \times q, \mathbf{C})$  (see Theorem 3.9).

For general references on bounded symmetric domains and their geometric properties, see [S], and Part III in [Fal]. For explicit calculations related to our example, see [P] and [H].

## 1. GEOMETRIC SETTING

Let  $p, q$  be two integers with  $1 \leq q \leq p$ , and let

$$(1) \quad D = \{z \in \text{Mat}(p \times q, \mathbf{C}) \mid \mathbf{1}_q - z^*z \gg 0\}.$$

Let  $G = \text{SU}(p, q) \subset \text{GL}(p + q, \mathbf{C})$ . An element  $g \in \text{GL}(p + q, \mathbf{C})$  will often be written as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where

$$a \in \text{Mat}(p \times p, \mathbf{C}), \quad b \in \text{Mat}(p \times q, \mathbf{C}), \quad c \in \text{Mat}(q \times p, \mathbf{C}), \quad d \in \text{Mat}(q \times q, \mathbf{C}).$$

In this notation, the conditions for  $g$  to belong to  $\text{U}(p, q, \mathbf{C})$  can be written as

$$(2) \quad \begin{aligned} a^*a - c^*c &= \mathbf{1}_p \\ b^*a - d^*c &= 0 \\ d^*d - b^*b &= \mathbf{1}_q. \end{aligned}$$

Define an action of the group  $\text{GL}(p + q, \mathbf{C})$  on  $\text{Mat}(p \times q, \mathbf{C})$  by

$$(3) \quad g(z) = (az + b)(cz + d)^{-1}.$$

The action is not everywhere defined, but it is certainly defined if  $g \in G$  and  $z \in D$ . It defines an action of  $G$  on  $D$ , and  $G$  (or rather  $\text{PSU}(p, q)$ ) is the neutral component of the group of all biholomorphic transformations of  $D$ .

The stabilizer of the base point  $0 \in D$  is the maximal compact subgroup  $K = S(\text{U}(p) \times \text{U}(q))$ . Its complexification is the complex group  $K^{\mathbf{C}} = S(\text{GL}(p, \mathbf{C}) \times \text{GL}(q, \mathbf{C}))$ . We also define the following subgroups

$$P^+ = \left\{ \begin{pmatrix} \mathbf{1}_p & z \\ 0 & \mathbf{1}_q \end{pmatrix}, z \in \text{Mat}(p \times q, \mathbf{C}) \right\}$$

$$P^- = \left\{ \begin{pmatrix} \mathbf{1}_p & 0 \\ w & \mathbf{1}_q \end{pmatrix}, w \in \text{Mat}(q \times p, \mathbf{C}) \right\}.$$

The corresponding *Harish Chandra decomposition* is the following identity

$$(4) \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \mathbf{1}_p & bd^{-1} \\ 0 & \mathbf{1}_q \end{pmatrix} \begin{pmatrix} a - bd^{-1}c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} \mathbf{1}_p & 0 \\ d^{-1}c & \mathbf{1}_q \end{pmatrix}$$

valid for  $g \in \text{GL}(p + q, \mathbf{C})$  if  $d$  is invertible.

The *automorphy kernel*  $k(z, w)$  is defined for  $z, w \in \text{Mat}(p \times q, \mathbf{C})$  wherever it makes sense by the formula

$$(5) \quad k(z, w) = (\mathbf{1}_q - w^*z)^{-1}.$$

In particular it is always well defined for  $z, w \in D$  and has values in  $\text{GL}(q, \mathbf{C})$ . It has the following law of transformation for  $g \in G$

$$(6) \quad k(g(z), g(w)) = j(g, z)k(z, w)j(g, w)^*,$$

where

$$(7) \quad j(g, z) = cz + d.$$

The *Shilov boundary* of  $D$  is the unitary Stiefel manifold  $S$  defined by

$$(8) \quad S = \{ \sigma \in \text{Mat}(p \times q, \mathbf{C}) \mid \sigma^* \sigma = \mathbf{1}_q \}.$$

The action of  $G$  extends to  $S$ , and it is clearly transitive on  $S$ . In fact the action of  $K$  is already transitive.

To go further, we need to make a specific choice of a base point in  $S$ . For this we first systematically write elements in  $\text{Mat}(p \times q, \mathbf{C})$  as

$$z = \begin{pmatrix} z_q \\ z' \end{pmatrix}$$

where  $z_q \in \text{Mat}(q \times q, \mathbf{C})$  and  $z' \in \text{Mat}((p - q) \times q, \mathbf{C})$ . With this convention, let  $ie = \begin{pmatrix} i\mathbf{1}_q \\ 0 \end{pmatrix}$  be the base point in  $S$ . Associated to this choice is the Cayley transform  $c$ , given by

$$z = \begin{pmatrix} z_q \\ z' \end{pmatrix} \mapsto c(z) = \begin{pmatrix} w_q \\ w' \end{pmatrix}$$

with

$$(9) \quad \begin{aligned} w_q &= (z_q + i\mathbf{1}_q)(iz_q + \mathbf{1}_q)^{-1} \\ w' &= -z'(iz_q + \mathbf{1}_q)^{-1}. \end{aligned}$$

The inverse of the Cayley transform is the map which to  $\begin{pmatrix} w_q \\ w' \end{pmatrix}$  associates the matrix  $\begin{pmatrix} z_q \\ z' \end{pmatrix}$  given by

$$(10) \quad \begin{aligned} z_q &= (iw_q - \mathbf{1}_q)^{-1}(i\mathbf{1}_q - w_q) \\ z' &= 2w'(iw_q - \mathbf{1}_q)^{-1}. \end{aligned}$$

The Cayley transform is a rational map, well defined on  $D$ . The image of  $D$  is the *Siegel domain of type II* defined by

$$(11) \quad {}^cD = \left\{ \begin{pmatrix} w_q \\ w' \end{pmatrix}, \frac{1}{2i}(w_q - w_q^*) - w'^*w' \gg 0 \right\}$$

and the image of the Shilov boundary (more exactly the part of the Shilov boundary where the Cayley transform is defined) is

$$(12) \quad {}^cS = \left\{ \begin{pmatrix} w_q \\ w' \end{pmatrix}, \frac{1}{2i}(w_q - w_q^*) = w'^*w' \right\}.$$

To the data

$$w_0 \in \text{Mat}((p-q) \times q, \mathbf{C})$$

$$h \in \text{GL}(q, \mathbf{C}), u \in \text{U}(p-q, \mathbf{C}), \text{ such that } \det h = (\det u)^{-1}$$

$$s \in \text{Herm}(q, \mathbf{C})$$

we associate the transform

$$(13) \quad \begin{aligned} w_q &\mapsto h^*w_qh + s + 2iw_0^*uw'h + iw_0^*w_0 \\ w' &\mapsto uw'h + w_0. \end{aligned}$$

Any such transform maps  ${}^cD$  in a one-to-one fashion into itself. These transforms form a group and it is exactly the group of affine holomorphic transforms of the domain  ${}^cD$ .

Let  $B$  be the stabilizer of the point  $ie$  in  $G$ . The conjugate group under the Cayley transform is  ${}^cB = c \circ B \circ c^{-1}$  and it turns out to be exactly the group of affine transforms of  ${}^cD$  we just described. Observe that the group  ${}^cB$  is transitive on  ${}^cD$  and on  ${}^cS$ .