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## A TRIPLE RATIO ON THE UNITARY STIEFEL MANIFOLD

by Jean-Louis CLERC

ABSTRACT. For the unitary Stiefel manifold  $S$  realized as the Shilov boundary of the unit ball  $D$  in  $\text{Mat}(p \times q, \mathbf{C})$ , we construct characteristic invariants for the (generic) orbits of the conformal group  $\mathbf{PSU}(p, q)$  in  $S \times S \times S$ . The construction uses the automorphy kernel of the bounded symmetric domain.

### INTRODUCTION

Let  $D = G/K$  be a bounded symmetric domain in a complex vector space  $\mathbf{C}^N$ , and let  $S$  be its Shilov boundary. The action of  $G$  extends to  $S$  and this action is transitive on  $S$ . It is generally referred to in the literature as the *conformal action* of  $G$  on  $S$ . One can show that the action is almost 2-transitive in the sense that  $G$  has a dense open orbit in  $S \times S$ . Hence it is a natural question to look for the  $G$ -orbits in  $S \times S \times S$  and for characteristic invariants of this action. If  $D$  happens to be of tube type (in which case  $\dim_{\mathbf{R}} S = \dim_{\mathbf{C}} D$ ), this question was solved in [CØ]. There are a finite number of open orbits in  $S \times S \times S$ , and the (generalized) *Maslov index* we constructed is a characteristic invariant for the  $G$ -action. In the case of the unit ball in  $\mathbf{C}^2$ , the Shilov boundary coincides with the topological boundary, namely the unit sphere  $S = \mathbf{S}^3$ . In [Ca], E. Cartan constructed a (real-valued) invariant for triples on  $S$  (he called  $S$  the “hypersphere”). Independently (and more than 50 years later) Korányi and Reimann studied the case of the unit ball in  $\mathbf{C}^n$  (see [KR]). Through the Cayley transform, the problem is changed into an equivalent problem for the Heisenberg group  $\mathbf{H}_n$  under the action of its conformal group  $G = \mathbf{PSU}(n + 1, 1)$ . For this situation, they studied a complex cross ratio on  $\mathbf{H}_n$ , from which they were able (in a rather indirect way) to construct a (real-valued) invariant for triples, which characterizes the  $G$ -orbits of triples in  $\mathbf{H}_n$ . Here we solve the problem for the case where  $D$

is the unit ball in the matrix space  $\text{Mat}(p \times q, \mathbf{C})$ ,  $S$  is the unitary Stiefel manifold  $\mathbf{S}_{p,q}$  and  $G = \mathbf{PSU}(p, q)$ . The invariant we construct for triples is of matrix-valued nature (it is a conjugacy class) and we give two versions of it (see Theorems 4.3 and 4.4). The basic strategy is to approach the Shilov boundary from inside. The (matrix-valued) *automorphy kernel* for the domain  $D$  is used to build a kernel for triples of points inside  $D$  which transforms nicely under the action of  $G$ . It remains to look carefully at the boundary behaviour of the kernel when the points approach the Shilov boundary  $S$ . This is only possible for triples satisfying a generic condition called *transversality* (see Proposition 2.1 for a definition). The *Cayley transform* plays an important role in the proofs. Finally the problem is reduced to a *linear* problem, which is related to the description of some orbits for the action  $(g, X) \mapsto gXg^*$  of  $\text{GL}_q$  on  $\text{Mat}(q \times q, \mathbf{C})$  (see Theorem 3.9).

For general references on bounded symmetric domains and their geometric properties, see [S], and Part III in [Fal]. For explicit calculations related to our example, see [P] and [H].

## 1. GEOMETRIC SETTING

Let  $p, q$  be two integers with  $1 \leq q \leq p$ , and let

$$(1) \quad D = \{z \in \text{Mat}(p \times q, \mathbf{C}) \mid \mathbf{1}_q - z^*z \gg 0\}.$$

Let  $G = \text{SU}(p, q) \subset \text{GL}(p + q, \mathbf{C})$ . An element  $g \in \text{GL}(p + q, \mathbf{C})$  will often be written as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where

$$a \in \text{Mat}(p \times p, \mathbf{C}), \quad b \in \text{Mat}(p \times q, \mathbf{C}), \quad c \in \text{Mat}(q \times p, \mathbf{C}), \quad d \in \text{Mat}(q \times q, \mathbf{C}).$$

In this notation, the conditions for  $g$  to belong to  $\text{U}(p, q, \mathbf{C})$  can be written as

$$(2) \quad \begin{aligned} a^*a - c^*c &= \mathbf{1}_p \\ b^*a - d^*c &= 0 \\ d^*d - b^*b &= \mathbf{1}_q. \end{aligned}$$

Define an action of the group  $\text{GL}(p + q, \mathbf{C})$  on  $\text{Mat}(p \times q, \mathbf{C})$  by

$$(3) \quad g(z) = (az + b)(cz + d)^{-1}.$$