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A TRIPLE RATIO ON THE UNITARY STIEFEL MANIFOLD

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ABSTRACT. For the unitary Stiefel manifold S realized as the Shilov boundary of the unit ball D in $\mathrm{Mat}(p \times q, \mathbf{C})$, we construct characteristic invariants for the (generic) orbits of the conformal group $\mathbf{PSU}(p,q)$ in $S \times S \times S$. The construction uses the automorphy kernel of the bounded symmetric domain.

INTRODUCTION

Let $D = G/K$ be a bounded symmetric domain in a complex vector space $\mathbf{C}^N$, and let S be its Shilov boundary. The action of G extends to S and this action is transitive on S. It is generally referred to in the literature as the *conformal action* of G on S. One can show that the action is almost 2-transitive in the sense that G has a dense open orbit in $S \times S$. Hence it is a natural question to look for the G-orbits in $S \times S \times S$ and for characteristic invariants of this action. If D happens to be of tube type (in which case $\dim_{\mathbf{R}} S = \dim_{\mathbf{C}} D$), this question was solved in [CØ]. There are a finite number of open orbits in $S \times S \times S$, and the (generalized) *Maslov index* we constructed is a characteristic invariant for the G-action. In the case of the unit ball in $\mathbf{C}^2$, the Shilov boundary coincides with the topological boundary, namely the unit sphere $S = \mathbf{S}^3$. In [Ca], E. Cartan constructed a (real-valued) invariant for triples on S (he called S the "hypersphere"). Independently (and more than 50 years later) Korányi and Reimann studied the case of the unit ball in $\mathbf{C}^n$ (see [KR]). Through the Cayley transform, the problem is changed into an equivalent problem for the Heisenberg group $\mathbf{H}_n$ under the action of its conformal group $G = \mathbf{PSU}(n+1,1)$. For this situation, they studied a complex cross ratio on $\mathbf{H}_n$, from which they were able (in a rather indirect way) to construct a (real-valued) invariant for triples, which characterizes the G-orbits of triples in $\mathbf{H}_n$. Here we solve the problem for the case where D

is the unit ball in the matrix space $\mathrm{Mat}(p \times q, \mathbf{C})$, S is the unitary Stiefel manifold $\mathbf{S}_{p,q}$ and $G = \mathbf{P}\mathrm{SU}(p,q)$. The invariant we construct for triples is of matrix-valued nature (it is a conjugacy class) and we give two versions of it (see Theorems 4.3 and 4.4). The basic strategy is to approach the Shilov boundary from inside. The (matrix-valued) *automorphy kernel* for the domain D is used to build a kernel for triples of points inside D which transforms nicely under the action of G. It remains to look carefully at the boundary behaviour of the kernel when the points approach the Shilov boundary S. This is only possible for triples satisfying a generic condition called *tranversality* (see Proposition 2.1 for a definition). The *Cayley transform* plays an important role in the proofs. Finally the problem is reduced to a *linear* problem, which is related to the description of some orbits for the action $(g,X) \mapsto gXg^*$ of GL_q on $\mathrm{Mat}(q \times q, \mathbf{C})$ (see Theorem 3.9).

For general references on bounded symmetric domains and their geometric properties, see [S], and Part III in [Fal]. For explicit calculations related to our example, see [P] and [H].

1. Geometric setting

Let p, q be two integers with $1 \leq q \leq p$, and let

(1)
$$D = \{z \in \mathrm{Mat}(p \times q, \mathbf{C}) \mid \mathbf{1}_q - z^* z \gg 0\}\,.$$

Let $G = \mathrm{SU}(p,q) \subset \mathrm{GL}(p+q, \mathbf{C})$. An element $g \in \mathrm{GL}(p+q, \mathbf{C})$ will often be written as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where

$$a \in \mathrm{Mat}(p \times p, \mathbf{C}),\ b \in \mathrm{Mat}(p \times q, \mathbf{C}),\ c \in \mathrm{Mat}(q \times p, \mathbf{C}),\ d \in \mathrm{Mat}(q \times q, \mathbf{C})\,.$$

In this notation, the conditions for g to belong to $\mathrm{U}(p,q,\mathbf{C})$ can be written as

(2)
$$\begin{aligned} a^*a - c^*c &= \mathbf{1}_p \\ b^*a - d^*c &= 0 \\ d^*d - b^*b &= \mathbf{1}_q\,. \end{aligned}$$

Define an action of the group $\mathrm{GL}(p+q, \mathbf{C})$ on $\mathrm{Mat}(p \times q, \mathbf{C})$ by

(3)
$$g(z) = (az + b)(cz + d)^{-1}\,.$$

The action is not everywhere defined, but it is certainly defined if $g \in G$ and $z \in D$. It defines an action of G on D, and G (or rather $\mathbf{PSU}(p,q)$) is the neutral component of the group of all biholomorphic transformations of D.

The stabilizer of the base point $0 \in D$ is the maximal compact subgroup $K = S(\mathrm{U}(p) \times \mathrm{U}(q))$. Its complexification is the complex group $K^{\mathbf{C}} = S(\mathrm{GL}(p,\mathbf{C}) \times \mathrm{GL}(q,\mathbf{C}))$. We also define the following subgroups

$$P^+ = \left\{ \begin{pmatrix} \mathbf{1}_p & z \\ 0 & \mathbf{1}_q \end{pmatrix},\ z \in \mathrm{Mat}(p \times q, \mathbf{C}) \right\}$$

$$P^- = \left\{ \begin{pmatrix} \mathbf{1}_p & 0 \\ w & \mathbf{1}_q \end{pmatrix},\ w \in \mathrm{Mat}(q \times p, \mathbf{C}) \right\}.$$

The corresponding *Harish Chandra decomposition* is the following identity

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \mathbf{1}_p & bd^{-1} \\ 0 & \mathbf{1}_q \end{pmatrix} \begin{pmatrix} a - bd^{-1}c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} \mathbf{1}_p & 0 \\ d^{-1}c & \mathbf{1}_q \end{pmatrix} \tag{4}$$

valid for $g \in \mathrm{GL}(p+q, \mathbf{C})$ if d is invertible.

The *automorphy kernel* $k(z,w)$ is defined for $z, w \in \mathrm{Mat}(p \times q, \mathbf{C})$ wherever it makes sense by the formula

$$k(z,w) = (\mathbf{1}_q - w^* z)^{-1}\,. \tag{5}$$

In particular it is always well defined for $z, w \in D$ and has values in $\mathrm{GL}(q,\mathbf{C})$. It has the following law of transformation for $g \in G$

$$k(g(z), g(w)) = j(g,z)\, k(z,w)\, j(g,w)^*\,, \tag{6}$$

where

$$j(g,z) = cz + d\,. \tag{7}$$

The *Shilov boundary* of D is the unitary Stiefel manifold S defined by

$$S = \left\{ \sigma \in \mathrm{Mat}(p \times q, \mathbf{C}) \mid \sigma^* \sigma = \mathbf{1}_q \right\}. \tag{8}$$

The action of G extends to S, and it is clearly transitive on S. In fact the action of K is already transitive.

To go further, we need to make a specific choice of a base point in S. For this we first systematically write elements in $\mathrm{Mat}(p \times q, \mathbf{C})$ as

$$z = \begin{pmatrix} z_q \\ z' \end{pmatrix}$$

where $z_q \in \mathrm{Mat}(q \times q, \mathbf{C})$ and $z' \in \mathrm{Mat}\left((p-q) \times q, \mathbf{C}\right)$. With this convention, let $ie = \begin{pmatrix} i\mathbf{1}_q \\ 0 \end{pmatrix}$ be the base point in S. Associated to this choice is the Cayley transform c, given by

$$z = \begin{pmatrix} z_q \\ z' \end{pmatrix} \longmapsto c(z) = \begin{pmatrix} w_q \\ w' \end{pmatrix}$$

with

(9)
$$\begin{aligned} w_q &= (z_q + i\mathbf{1}_q)(iz_q + \mathbf{1}_q)^{-1} \\ w' &= -z'(iz_q + \mathbf{1}_q)^{-1} \,. \end{aligned}$$

The inverse of the Cayley transform is the map which to $\begin{pmatrix} w_q \\ w' \end{pmatrix}$ associates the matrix $\begin{pmatrix} z_q \\ z' \end{pmatrix}$ given by

(10)
$$\begin{aligned} z_q &= (iw_q - \mathbf{1}_q)^{-1}(i\mathbf{1}_q - w_q) \\ z' &= 2w'(iw_q - \mathbf{1}_q)^{-1} \,. \end{aligned}$$

The Cayley transform is a rational map, well defined on D. The image of D is the *Siegel domain of type II* defined by

(11)
$$^cD = \left\{ \begin{pmatrix} w_q \\ w' \end{pmatrix} , \frac{1}{2i}(w_q - w_q^*) - w'^*w' \gg 0 \right\}$$

and the image of the Shilov boundary (more exactly the part of the Shilov boundary where the Cayley transform is defined) is

(12)
$$^cS = \left\{ \begin{pmatrix} w_q \\ w' \end{pmatrix} , \frac{1}{2i}(w_q - w_q^*) = w'^*w' \right\} .$$

To the data

$$\begin{aligned} &w_0 \in \mathrm{Mat}((p-q) \times q, \mathbf{C}) \\ &h \in \mathrm{GL}(q, \mathbf{C}),\ u \in \mathrm{U}(p-q, \mathbf{C}),\ \text{such that } \det h = (\det u)^{-1} \\ &s \in \mathrm{Herm}(q, \mathbf{C}) \end{aligned}$$

we associate the transform

(13)
$$\begin{aligned} w_q &\longmapsto h^* w_q h + s + 2i w_0^* u w' h + i w_0^* w_0 \\ w' &\longmapsto u w' h + w_0 \,. \end{aligned}$$

Any such transform maps ^cD in a one-to-one fashion into itself. These transforms form a group and it is exactly the group of affine holomorphic transforms of the domain ^cD.

Let B be the stabilizer of the point ie in G. The conjugate group under the Cayley transform is $^cB = c \circ B \circ c^{-1}$ and it turns out to be exactly the group of affine transforms of ^cD we just described. Observe that the group ^cB is transitive on ^cD and on ^cS.

2. ACTION OF G ON $S \times S$ AND $S \times S \times S$

We now study the action of G on pairs of points of S. The main notion to be introduced is *transversality*, a notion that could be defined for any bounded symmetric domain. We give several equivalent definitions for our case.

PROPOSITION 2.1. *Let σ and ξ be two elements of S. Then the following are equivalent:*

(i) $\det(\mathbf{1}_q - \xi^*\sigma) \neq 0$;

(ii) $\xi - \sigma$ *injective;*

(iii) $\det(\mathbf{1}_p - \xi\sigma^*) \neq 0$.

If one of these equivalent conditions is satisfied, then σ and ξ are said to be transverse.

Proof. Assume (i). As $\mathbf{1}_q = \xi^*\xi$, this condition amounts to $\det(\xi^*(\xi-\sigma)) \neq 0$, which in particular shows that $\xi-\sigma$ is injective. Conversely, assume $\xi-\sigma$ is injective and let $v \in \mathbf{C}^q$ be such that $v = \xi^*\sigma v$. Now

$$\|v\| = \|\xi^*\sigma v\| \leq \|\sigma v\| \leq \|v\|,$$

and hence $\|\xi^*\sigma v\| = \|\sigma v\|$, which is possible only if $\sigma v \in \operatorname{Im}\xi$. So there exists $w \in \mathbf{C}^q$, such that $\sigma v = \xi w$. But taking the image of both sides by ξ^* yields $v = w$, and hence $\sigma v = \xi v$, so that $v = 0$. So $\mathbf{1}_q - \xi^*\sigma$ is injective and hence (ii) $\Longrightarrow$ (i). Under the same assumption (ii), let us prove that $\xi\sigma^*$ cannot have 1 as an eigenvalue. Suppose $v \in \mathbf{C}^p$ is such that $\xi\sigma^* v = v$. As ξ is a partial isometry, this forces $\|\sigma^* v\| = \|v\|$, and hence v belongs to the image of the map σ, so there exists $w \in \mathbf{C}^q$ such that $v = \sigma w$. But then we also have $v = \xi\sigma^*\sigma w = \xi w$ and hence $(\sigma - \xi)w = 0$ which forces $w = 0$. Hence (iii) follows from (ii). Finally assume (iii). Then as σ is injective, $(\mathbf{1}_p - \xi\sigma^*) \circ \sigma = \sigma - \xi$ is also injective. Hence (iii) $\Longrightarrow$ (ii). $\square$

We will use the notation $\sigma \top \xi$ to denote transversality. It is a symmetric condition. It is invariant under the action of G, as can easily be concluded from (6). For $\sigma \in S$, let

$$S^{\sigma}_{\top} = \{\xi \mid \sigma \top \xi\}.$$

Observe that the set $S^{ie}_{\top}$ is exactly the subset in S where the Cayley transform is defined.

Let

(14) $$S^2_{\top} = \{(\sigma,\xi) \in S \times S \mid \sigma \top \xi\}\,.$$

As base point in $S^2_{\top}$ we choose $(ie, -ie)$. Observe that $c(-ie) = 0$.

THEOREM 2.2. *The group G acts transitively on $S^2_{\top}$.*

Proof. Let $(\sigma,\xi) \in S^2_{\top}$ and let us show that there exists an element of G which maps (σ,ξ) to $(ie, -ie)$. As G is transitive on S, we may assume that $\sigma = ie$. Then the transversality condition shows that ξ belongs to the domain of the Cayley transform. The element $c(\xi)$ belongs to ${}^cS$, and we have already noticed that ${}^cB$ is transitive on ${}^cS$. Hence $c(\xi)$ can be mapped to $0 = c(-ie)$. Taking the image under the inverse Cayley transform gives the result. $\square$

Denote by L the stabilizer of the base point $(ie, -ie)$ in B. Under a Cayley transform, the group ${}^cL = c \circ L \circ c^{-1}$ is the stabilizer in ${}^cB$ of the element 0. Hence it is the subgroup of linear transformations given by

$$\begin{aligned} w_q &\longmapsto h^* w_q h \\ w' &\longmapsto uwh \end{aligned}$$

where $h \in \mathrm{GL}(q, \mathbf{C})$, $u \in \mathrm{U}(p-q)$ and $\det h = (\det u)^{-1}$.

LEMMA 2.3. *Let $\begin{pmatrix} w_q \\ w' \end{pmatrix}, \begin{pmatrix} v_q \\ v' \end{pmatrix} \in {}^cS$. Then they belong to the same orbit under the action of ${}^cL$ if and only if w_q and v_q belong to the same orbit under the action of* $\mathrm{GL}(q, \mathbf{C})$.

Proof. One implication being trivial, we only have to prove the other one. So assume there exists $h \in \mathrm{GL}(q, \mathbf{C})$ such that $v_q = h^* w_q h$. Let μ be a complex number such that $\mu^{p-q} = \det h$ and let $u = \mu^{-1}\mathbf{1}_{p-q}$. Clearly $(\det u)^{-1} = \det h$. Using the action of (h, u) we may assume that $v_q = w_q$. Let $s_q = \frac{1}{2i}(w_q - w_q^*)$. This is an Hermitian matrix and as w_q and v_q belong to ${}^cS$, we get

$$s_q = w'^* w' = v'^* v'\,.$$

Looking to the columns of w' (or v'), we may think of w' as a family of q vectors in $\mathbf{C}^{p-q}$. Then the matrix s_q is the Gram matrix of these vectors. But two sets of vectors in $\mathbf{C}^{p-q}$ are conjugate under the action of the unitary group $\mathrm{U}(p-q)$ if and only if they have the same Gram matrix. Hence there exists $u \in \mathrm{U}(p-q)$ such that $v' = uw'$. Let λ be a complex number such that $\lambda^q = \det u$. Then using the action of $(\lambda^{-1}\mathbf{1}_q, u)$, we get the result. $\square$

Let us denote by H_q the real vector space of $q \times q$ Hermitian matrices, and let Ω_q be the subset of all positive-definite matrices. For any integer r such that $0 \leq r \leq q$ let $\Omega_q^{(r)}$ be the set of all positive semi-definite $q \times q$ Hermitian matrices of rank less than r. For $r < q$, the set $\Omega_q^{(r)}$ is contained in the boundary of Ω_q, whereas for $r = q$, $\Omega_q^{(q)} = \overline{\Omega}_q$.

Let

$$T_q^{(r)} = \{x + iy \mid x \in H_q, y \in \Omega_q^{(r)}\}\,.$$

The group $\mathrm{GL}(q, \mathbf{C})$ acts on $T_q^{(r)}$ by the action $(h, w) \longmapsto hwh^*$.

Finally let

$$\widetilde{T}_q^{(r)} = \{z \in T_q^{(r)} \mid z \text{ invertible}\}\,.$$

Clearly the action of $\mathrm{GL}(q, \mathbf{C})$ preserves $\widetilde{T}_q^{(r)}$.

Let $\begin{pmatrix} w_q \\ w' \end{pmatrix}$ be in ${}^cS$. Then $w_q = x_q + iw'^*w'$, with $x_q \in H_q$. Let

$$r = \inf(q, p - q)\,.$$

The rank of the matrix w'^*w' is at most r. Hence w_q belongs to $T_q^{(r)}$. Conversely, it is easily seen that any positive semi-definite Hermitian matrix of rank at most r can be written as w'^*w' for some $w' \in \mathrm{Mat}((p-q) \times q, \mathbf{C})$.

Let

$$S_\top^3 = \{(\sigma_1, \sigma_2, \sigma_3) \in S \times S \times S \mid \sigma_1 \top \sigma_2, \sigma_2 \top \sigma_3, \sigma_3 \top \sigma_1\}\,. \tag{15}$$

THEOREM 2.4. *The G-orbits in $S_\top^3$ are in one-to-one correspondance with the orbits of* $\mathrm{GL}(q, \mathbf{C})$ *in* $\widetilde{T}_q^{(r)}$.

Proof. From Theorem 2.2 we already know that any orbit contains an element of the form $(ie, -ie, \sigma)$ with $\sigma \in S$. Now use the Cayley transform. The element $w = c(\sigma)$ is in ${}^cS$, and the transversality condition is equivalent to the condition $\det(w_q) \neq 0$. In other words, $w_q \in \widetilde{T}_q^{(r)}$. The result now follows from Lemma 2.3. $\square$

3. Orbits for the GL_q-action on $\widetilde{T}_q$

Any $z \in \mathrm{Mat}(q \times q, \mathbf{C})$ can be written in a unique way as $z = x + iy$, with $x, y \in H_q$. We will be concerned with the set $\widetilde{T}_q$ defined by

(16) $\widetilde{T}_q = \{z \in \mathrm{Mat}(q \times q, \mathbf{C}) \mid z = x + iy,\ x \in H_q,\ y \in \overline{\Omega}_q,\ \det z \neq 0\}$.

Its interior is the classical *tube domain* over the cone Ω_q, namely

$$T_q = \{z \in \mathrm{Mat}(q \times q, \mathbf{C}) \mid z = x + iy,\ y \in \Omega_q\}.$$

Let $G = \mathrm{GL}(q, \mathbf{C})$ act on $\mathrm{Mat}(q \times q, \mathbf{C})$ by

(17) $$(g, z) \longmapsto gzg^*.$$

The spaces $H_q, \Omega_q, \overline{\Omega}_q$ are stable under this action, and hence $\widetilde{T}_q$ and T_q are invariant subsets under this action. We investigate the orbits and describe a full set of invariants for this action.

There is a natural invariant associated to a $\mathrm{GL}(q, \mathbf{C})$-orbit. To any $z \in \widetilde{T}_q$, we associate its *angular matrix* defined by

(18) $$a = a(z) = {z^*}^{-1} z.$$

Then the matrix associated to gzg^* is ${g^*}^{-1} a g^*$, so that the angular matrix $a(z)$ belongs to the same conjugacy class when z runs through a $\mathrm{GL}(q, \mathbf{C})$-orbit. As we shall see (Theorem 3.3 and Theorem 3.13), this invariant is close to characterizing the orbits.

Let us first prove some elementary properties of the angular matrix.

PROPOSITION 3.1. *Let $z = x + iy \in \widetilde{T}_q$, and let $a = {z^*}^{-1} z$ be its angular matrix. Then*

(i) $\mathrm{Sp}(a) \subset \mathrm{U}_1 = \{\mu \in \mathbf{C}, |\mu| = 1\}$;

(ii) *if* $1 \in \mathrm{Sp}(a)$, *then y is degenerate and*

$$\{v \in \mathbf{C}^q \mid av = v\} = \{v \in \mathbf{C}^q \mid yv = 0\}.$$

Proof. Let μ be an eigenvalue of a, and let $v \neq 0$ be an eigenvector for the eigenvalue μ. Then $zv = \mu z^* v$, and hence

$$(zv, v) = \mu(z^* v, v) = \mu(v, zv) = \mu\,\overline{(zv, v)}.$$

If $(zv, v) \neq 0$, then $|\mu| = 1$. So we now assume $(zv, v) = 0$. This amounts to $(xv, v) + i(yv, v) = 0$, so that in particular $(yv, v) = 0$. Now recall that y is positive semi-definite. So the condition $(yv, v) = 0$ implies that $yv = 0$. From this it follows that $zv = xv = z^* v$, and as z is assumed to be invertible, this implies $\mu = 1$. This shows (i) and part of (ii). Conversely, the condition $yv = 0$ implies trivially $av = v$. $\square$

In particular, we may consider the polynomial $d(\mu) = \det(z - \mu z^*)$. The roots of d are the eigenvalues of the angular matrix. The set of these roots, counted with their multiplicities, will be called the *angular spectrum* of z.

We first consider the case of T_q. So let $z = x + iy \in T_q$. Then as y is positive-definite, we may define its square root $y^{1/2}$ as the unique positive-definite Hermitian matrix whose square is y. Then we may write

$$x + iy = y^{\frac{1}{2}}(y^{-\frac{1}{2}}xy^{-\frac{1}{2}} + i1_q)y^{\frac{1}{2}} .$$

This shows that any $\mathrm{GL}(q, \mathbf{C})$-orbit contains some element of the form $x+i1_q$, where $x \in H_q$. But by the classical diagonalization theorem for Hermitian forms, there exists an orthonormal basis in which the Hermitian form associated to x is diagonal. In other words, there exists a unitary matrix u and real numbers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_q$ such that

$$uxu^* = \Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_q \end{pmatrix} .$$

Moreover, if Λ and Λ' are two such diagonal matrices, then $\Lambda + i\mathbf{1}_q$ and $\Lambda' + i\mathbf{1}_q$ are not conjugate under the action of $\mathrm{GL}(q, \mathbf{C})$ unless $\Lambda = \Lambda'$. Hence we have shown the following result, which of course is the well-known fact that there is a simultaneous diagonalization for two Hermitian forms if one of them is positive-definite.

THEOREM 3.2. *The set of matrices of the form*

$$\Lambda = \begin{pmatrix} \lambda_1 + i & & & \\ & \lambda_2 + i & & \\ & & \ddots & \\ & & & \lambda_q + i \end{pmatrix} \tag{19}$$

with $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_q$ *is a full set of representatives of the* $\mathrm{GL}(q, \mathbf{C})$*-orbits in* T_q.

The angular matrix associated to Λ is

$$\begin{pmatrix} \frac{\lambda_1+i}{\lambda_1-i} & & & \\ & \frac{\lambda_2+i}{\lambda_2-i} & & \\ & & \ddots & \\ & & & \frac{\lambda_q+i}{\lambda_q-i} \end{pmatrix} . \tag{20}$$

The latter is a semi-simple matrix with spectral values

$$\mu_j = \frac{\lambda_j + i}{\lambda_j - i}$$

for $1 \leq j \leq q$. Observe that these spectral values are complex numbers of modulus 1, but always different from 1. From the u_j we may recover the λ_j by the formula

$$\lambda_j = i\,\frac{1+\mu_j}{1-\mu_j}\,.$$

From these observations we get the following result.

THEOREM 3.3. *Two elements z and z' of T_q belong to the same* $\mathrm{GL}(q,\mathbf{C})$*-orbit if and only if their angular matrices are conjugate. The angular spectrum is a full set of invariants for the action of* $\mathrm{GL}(q,\mathbf{C})$ *on* T_q.

The situation for $\widetilde{T}_q$ is more complicated. In fact we may consider the extreme case where $y=0$. Then x corresponds to a non-degenerate Hermitian form, and the orbit picture is given by the signature. So we need to consider matrices of the form

$$\Upsilon = \Upsilon_{n_+,n_-} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix}$$

with n_+ diagonal entries equal to $+1$ and n_- diagonal entries equal to -1, n_+ and n_- being arbitrary nonnegative integers such that $n_+ + n_- = q$. The corresponding angular matrix is the identity matrix $\mathbf{1}_q$.

Another source of difficulty comes from the fact that it is not always possible to find a basis in which both Hermitian forms associated to x and y are diagonal. For instance if $q=2$, consider the matrix

$$z = \begin{pmatrix} i & \frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix} + i\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Notice that its angular matrix is

$$a = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

which is not semisimple.

From these examples we see that neither the angular spectrum of z nor the conjugacy class of the angular matrix characterizes the orbit of z.

Let n_1, n_2, n_3, n_4 be four nonnegative integers such that $n_1+2n_2+n_3+n_4 = q$, and let $\lambda_1, \lambda_2, \dots, \lambda_{n_1}$ be n_1 real numbers satisfying the condition

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n_1}.$$

To such data we associate the matrix $\Lambda = \Lambda(\lambda_1, \lambda_2, \dots, \lambda_{n_1}, n_2, n_3, n_4)$ given by

$$\tag{21} \begin{pmatrix} \lambda_1+i &&&&&&&&&&&& \\ & \ddots &&&&&&&&&&& \\ && \lambda_{n_1}+i &&&&&&&&&& \\ &&& i & 1 &&&&&&&& \\ &&& 1 & 0 &&&&&&&& \\ &&&&& \ddots &&&&&&& \\ &&&&&& i & 1 &&&&& \\ &&&&&& 1 & 0 &&&&& \\ &&&&&&&& 1 &&&& \\ &&&&&&&&& \ddots &&& \\ &&&&&&&&&& 1 && \\ &&&&&&&&&&& -1 & \\ &&&&&&&&&&&& \ddots \\ &&&&&&&&&&&&& -1 \end{pmatrix}$$

where there are n_2 diagonal 2×2 submatrices of the form $\begin{pmatrix} i & 1 \\ 1 & 0 \end{pmatrix}$, n_3 diagonal terms equal to 1 and n_4 diagonal terms equal to -1.

THEOREM 3.4. *Any* $\mathrm{GL}(q, \mathbf{C})$ *orbit in* $\widetilde{T}_q$ *contains one and only one matrix of the form* $\Lambda(\lambda_1, \lambda_2, \dots, \lambda_{n_1}, n_2, n_3, n_4)$.

Before beginning the proof, let us prove a couple of lemmas. Lemmas 3.6 and 3.7 are related to the classical Gauss's algorithm for diagonalizing an Hermitian form. Let r, s, n be three nonnegative integers such that $r+s=n$.

LEMMA 3.5. *The stabilizer in* $\mathrm{GL}(n, \mathbf{C})$ *of the matrix* $y_r = \begin{pmatrix} \mathbf{1}_r & \\ & \mathbf{0}_s \end{pmatrix}$ *is the subgroup*

$$\tag{22} G_r = \left\{ \begin{pmatrix} u & v \\ 0 & h \end{pmatrix} \right\}$$

where $u \in \mathrm{U}(r)$, $v \in \mathrm{Mat}(r, s)$, $h \in \mathrm{GL}(s, \mathbf{C})$.

Proof. Easy computation.

Now we study the action of G_r in H_n. If $x \in H_n$, let us write

$$x = \begin{pmatrix} \alpha & b \\ b^* & \gamma \end{pmatrix}$$

where $\alpha \in H_r, b \in \mathrm{Mat}(r \times s, \mathbf{C})$ and $\gamma \in H_s$. If $g = \begin{pmatrix} u & v \\ 0 & h \end{pmatrix} \in G_r$, then $gxg^* = \begin{pmatrix} \alpha' & b' \\ b'^* & \gamma' \end{pmatrix}$, with

$$\alpha' = u\alpha u^* + ubv^* + vb^*u^* + v\gamma v^*$$
$$b' = ubh^* + v\gamma h^*$$
$$\gamma' = h\gamma h^* .$$

LEMMA 3.6. *Let* $x = \begin{pmatrix} \alpha & b \\ b^* & \gamma \end{pmatrix} \in H_n$, *with* $\alpha \in H_r$, $b \in \mathrm{Mat}(r \times s, \mathbf{C})$ *and* $\gamma \in H_s$. *Assume* $\det \gamma \neq 0$. *Then the orbit of* x *under* G_r *contains a matrix of the form* $\begin{pmatrix} \alpha' & 0 \\ 0 & \gamma \end{pmatrix}$ *with* $\alpha' \in H_r$.

Proof. This is a consequence of the previous formula with $u = \mathbf{1}_r$, $v = -b\gamma^{-1}$ and $h = \mathbf{1}_s$.

LEMMA 3.7. *Let* $x = \begin{pmatrix} \alpha & b \\ b^* & 0 \end{pmatrix} \in H_n$, *with* $\operatorname{rank} b = s$ *(so in particular* $r \geq s$*). Then the orbit of* x *under* G_r *contains an element of the form*

$$\begin{pmatrix} \beta & 0 & 0 \\ 0 & 0 & \mathbf{1}_s \\ 0 & \mathbf{1}_s & 0 \end{pmatrix}$$

with $\beta \in H_{r-s}$.

Proof. Consider the subgroup $\left\{ \begin{pmatrix} u & 0 \\ 0 & h \end{pmatrix},\ u \in \mathrm{U}(r),\ h \in \mathrm{GL}_s(\mathbf{C}) \right\}$. It acts on the component b by $b' = ubh^*$. As $\operatorname{rank}(b) = s$, we may think of b as a set of s independent vectors in $\mathbf{C}^r$. By the Gram-Schmidt process, it is possible to find $h \in \mathrm{GL}_s(\mathbf{C})$ such that bh^* is a s-orthonormal frame in $\mathbf{C}^r$. But now two such frames are conjugate by the (left) action of $\mathrm{U}(r)$. Hence there exists $u \in \mathrm{U}(r)$ such that

$$ubh^* = \begin{pmatrix} 0 \\ \mathbf{1}_s \end{pmatrix}.$$

The matrix x we started with is conjugate under G_r to a matrix of the form

$$\begin{pmatrix} \alpha' & c & 0 \\ c^* & \beta & \mathbf{1}_s \\ 0 & \mathbf{1}_s & 0 \end{pmatrix}$$

where $\alpha' \in H_{r-s}$, $\beta \in H_s$ and $c \in \mathrm{Mat}((r-s)\times s, \mathbf{C})$. Now we use the action of the element

$$g = \begin{pmatrix} \mathbf{1}_{r-s} & 0 & -c \\ 0 & \mathbf{1}_s & -\frac{\beta}{2} \\ 0 & 0 & \mathbf{1}_s \end{pmatrix} \in G_r$$

to get the result. $\square$

We are now ready to start the proof of Theorem 3.4.

STEP 1. Let $z = x + iy \in \widetilde{T}_q$. As y is positive semidefinite, there exists an element $g \in \mathrm{GL}(q, \mathbf{C})$ such that

$$gyg^* = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix},$$

with r diagonal entries equal to 1, and s diagonal entries equal to 0, r and s being nonnegative integers satisfying $r + s = q$. In other terms, any $\mathrm{GL}(q, \mathbf{C})$-orbit in $\widetilde{T}_q$ contains an element of the form

$$\begin{pmatrix} \alpha + i\mathbf{1}_r & b \\ b^* & \gamma \end{pmatrix}$$

with $\alpha \in H_r, \gamma \in H_s, b \in \mathrm{Mat}(r \times s, \mathbf{C})$.

STEP 2. Now assume x is of the form

$$x = \begin{pmatrix} \alpha + i\mathbf{1}_r & b \\ b^* & \gamma \end{pmatrix}.$$

Consider γ. It is an Hermitian matrix of size s, and under the action of $\mathrm{GL}(s, \mathbf{C})$ it can be transformed to

$$\begin{pmatrix} \mathbf{0}_{n_2} & 0 & 0 \\ 0 & \mathbf{1}_{n_3} & 0 \\ 0 & 0 & -\mathbf{1}_{n_4} \end{pmatrix}$$

where $n_2 + n_3 + n_4 = s$. Hence x is conjugate under the action of G_r to an element of the form

$$\begin{pmatrix} \alpha & b' & c' \\ b'^* & 0 & 0 \\ c'^* & 0 & \Upsilon \end{pmatrix}$$

where $\alpha \in H_r$, $b' \in \mathrm{Mat}(r \times n_2, \mathbf{C})$, $c' \in \mathrm{Mat}(r \times (n_3 + n_4), \mathbf{C})$ and

$$\Upsilon = \begin{pmatrix} \mathbf{1}_{n_3} & 0 \\ 0 & -\mathbf{1}_{n_4} \end{pmatrix}.$$

Using Lemma 3.6, we see that x is conjugate under the action of G_s to an element of the form

$$\begin{pmatrix} \alpha'' & b'' & 0 \\ b''^* & 0 & 0 \\ 0 & 0 & \Upsilon \end{pmatrix},$$

with $\alpha'' \in H_r$, $b'' \in \mathrm{Mat}(r \times n_2, \mathbf{C})$.

STEP 3. Assume now that

$$x = \begin{pmatrix} \alpha & b & 0 \\ b^* & 0 & 0 \\ 0 & 0 & \Upsilon \end{pmatrix}$$

with $\alpha \in H_r$ and $b \in \mathrm{Mat}(r \times n_2, \mathbf{C})$. Recall that

$$x + iy = \begin{pmatrix} \alpha + i\mathbf{1}_r & b & 0 \\ b^* & 0 & 0 \\ 0 & 0 & \Upsilon \end{pmatrix}$$

is assumed to be invertible. This shows that $\mathrm{rank}(b) = n_2$. So we may apply Lemma 3.7 to see that x is conjugate under G_r to an element of the form

$$\begin{pmatrix} \beta & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1}_{n_2} & 0 \\ 0 & \mathbf{1}_{n_2} & 0 & 0 \\ 0 & 0 & 0 & \Upsilon \end{pmatrix}$$

with $\beta \in H_{r-n_2}$.

STEP 4. Set $n_1 = r - n_2$. The last step is just to put the element $\beta \in H_{n_1}$ in diagonal form under the action of $\mathrm{U}(n_1)$. Up to minor rearrangements of the matrix, this shows that any $\mathrm{GL}(q, \mathbf{C})$-orbit in $\widetilde{T}_q$ contains an element of the form $\Lambda(\lambda_1, \ldots, \lambda_{n_1}, n_2, n_3, n_4)$.

STEP 5. It remains to show that two Λ's are not conjugate under $\mathrm{GL}(q,\mathbf{C})$. The angular matrix associated to $\Lambda(\lambda_1,\dots,\lambda_{n_1},n_2,n_3,n_4)$ is

$$\begin{pmatrix} \frac{\lambda_1+i}{\lambda_1-i} & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & \frac{\lambda_{n_1}+i}{\lambda_{n_1}-i} & & & & & & & \\ & & & 1 & 0 & & & & & \\ & & & 2i & 1 & & & & & \\ & & & & & \ddots & & & & \\ & & & & & & 1 & 0 & & \\ & & & & & & 2i & 1 & & \\ & & & & & & & & 1 & \\ & & & & & & & & & \ddots \\ & & & & & & & & & & 1 \end{pmatrix}$$

where there are n_2 2×2 submatrices $\begin{pmatrix} 1 & 0 \\ 2i & 1 \end{pmatrix}$, and n_3+n_4 diagonal elements equal to 1. From the Jordan normal form theorem, we deduce that if $\Lambda(\lambda_1,\dots,\lambda_{n_1},n_2,n_3,n_4)$ and $\Lambda(\lambda'_1,\dots,\lambda'_{n_1},n'_2,n'_3,n'_4)$ are in a same $\mathrm{GL}(q,\mathbf{C})$-orbit, then $n_1=n'_1$, $\lambda_j=\lambda'_j$ for all $j, 1\le j\le n_1$, $n_2=n'_2$ and $n_3+n_4=n'_3+n'_4$. Now the matrix $\Lambda(\lambda_1,\dots,\lambda_{n_1},n_2,n_3,n_4)=L+iM$ and $\Lambda'=L'+iM'$, with $L,L',M,M'\in H_n$. As Λ and Λ' are supposed to be in the same $\mathrm{GL}(q,\mathbf{C})$-orbit, L and L' are also in the same $\mathrm{GL}(q,\mathbf{C})$-orbit, and so they must have the same signature. This forces $n_3=n'_3$ and $n_4=n'_4$, and hence $\Lambda=\Lambda'$.

We can now give the solution to the orbit problem we addressed at the end of Section 2. Recall that for any integer r such that $0\le r\le q$ we defined

$$\widetilde{T}_q^{(r)}=\{z=x+iy \mid y\in\overline{\Omega}_q,\ \operatorname{rank}(y)\le r,\ z \text{ invertible}\}.$$

LEMMA 3.8. *Let n_1,n_2,n_3,n_4 be four integers such that*

$$n_1+2n_2+n_3+n_4=q,$$

and let $\lambda_1,\dots,\lambda_{n_1}$ be n_1 real numbers. Then the standard matrix $\Lambda=\Lambda(\lambda_1,\dots,\lambda_{n_1},n_2,n_3,n_4)$ belongs to $\widetilde{T}_q^{(r)}$ if and only if $n_1+n_2\le r$.

In fact the rank of $\frac{1}{2i}(\Lambda-\Lambda^*)$ is n_1+n_2.

THEOREM 3.9. *Any* $\mathrm{GL}(q,\mathbf{C})$*-orbit in* $\widetilde{T}_q^{(r)}$ *contains a unique standard matrix* $\Lambda((\lambda_1,\dots,\lambda_{n_1},n_2,n_3,n_4)$ *with* $n_1+n_2\le r$.

We now want an analog of Theorem 3.3. As we have already noticed, the conjugacy class of the angular matrix does not determine the orbit of the matrix. We need a finer invariant, which we will construct now.

LEMMA 3.10. *The space* $\widetilde{T}_q$ *is connected and simply connected.*

Proof. As T_q is connected and $T_q\subset\widetilde{T}_q\subset\overline{T_q}$, the space $\widetilde{T}_q$ is connected. Take $i\mathbf{1}_q$ as base point in $\widetilde{T}_q$, and observe that for any $z\in\widetilde{T}_q$ and any $s>0$, $z+is\mathbf{1}_q$ is in T_q. So if $\big(\gamma(t),t\in[0,1]\big)$ is a path in $\widetilde{T}_q$ starting and ending at $i\mathbf{1}_q$ then we can deform it by homotopy to $\gamma_s(t)=\gamma(t)+is(s-1)\mathbf{1}_q$, which for $s>0$ is a path inside T_q. But T_q as a tube-type domain is simply connected. $\square$

The function $z\mapsto\det(z)$ is a continuous function from $\widetilde{T}_q$ into $\mathbf{C}^*$. From Lemma 3.10, there exists a unique continuous determination of the argument of $\det(z)$ denoted by $\arg\det\colon\widetilde{T}_q\longrightarrow\mathbf{R}$ such that $\arg\det i\mathbf{1}_q=q\frac{\pi}{2}$. If $Y\in\Omega_q$, then $\arg\det iy=q\frac{\pi}{2}$. If $z\in\widetilde{T}_q$ and $g\in\mathrm{GL}(q,\mathbf{C})$, then $\det gzg^*=|\det g|^2\det z$, and $gi\mathbf{1}_qg^*=igg^*\in i\Omega_q$, so that

$$\arg\det gzg^*=\arg\det z\,.$$

This provides a new invariant for the action of $\mathrm{GL}(q,\mathbf{C})$ on $\widetilde{T}_q$.

LEMMA 3.11. *Let* $\Lambda=\Lambda(\lambda_1,\dots,\lambda_{n_1},n_2,n_3,n_4)$. *Then*

$$\arg\det\Lambda=\arg(\lambda_1+i)+\cdots+\arg(\lambda_{n_1}+i)+n_2\pi+n_4\pi \tag{23}$$

where arg *is used for the principal determination of the argument of a non-zero complex number.*

Proof. We need to describe a continuous path from $i\mathbf{1}_q$ to Λ inside $\widetilde{T}_q$. For clarity of exposition, we describe successively the path for each diagonal block (either a one-dimensional or a two-dimensional submatrix) of Λ, and compute the contribution of each block to the function $\arg\det$.

For a block of the form $\lambda+i$, with $\lambda\in\mathbf{R}$ we use the path $t\mapsto t\lambda+i$, $0\le t\le 1$, and so the contribution of this block is $\arg(\lambda+i)$.

For a block of the form $\begin{pmatrix} i & 1 \\ 1 & 0 \end{pmatrix}$, we use the path

$$t \mapsto \begin{pmatrix} i & t \\ t & i(1-t^2) \end{pmatrix}, \; 0 \leq t \leq 1 .$$

The corresponding determinant of this 2×2-block is constant along the path and equal to -1. Hence the contribution of this block is $2\frac{\pi}{2} = \pi$.

For a block of the form 1, we use the path $t \mapsto e^{i\frac{\pi}{2}(1-t)}$, $0 \leq t \leq 1$, and we see that the corresponding contribution is 0.

For a block of the form -1, we use the path $t \mapsto e^{i\frac{\pi}{2}(1+t)}$, $0 \leq t \leq 1$, and we see that the corresponding contribution is π.

Putting together the contribution of the blocks, we get the result. □

COROLLARY 3.12. *Let Λ and Λ' be two standard matrices. Assume that their angular matrices coincide and that* $\arg\det\Lambda = \arg\det\Lambda'$. *Then* $\Lambda = \Lambda'$.

Proof. In fact we noticed that the equality of angular matrices implies the equality of the parameters except for $n_3 = n_3'$ and $n_4 = n_4'$. But from (23), we see that the equality of the determination of the arguments of the determinants implies $n_4 = n_4'$ (and hence $n_3 = n_3'$). □

Now we can state the conclusion of this section, which is a consequence of Theorem 3.4 and Corollary 3.12.

THEOREM 3.13. *Let $z, z' \in \widetilde{T}_q$, and assume that the angular matrices of z and z' are conjugate, and that* $\arg\det z = \arg\det z'$. *Then z and z' belong to the same orbit under the action of* $\mathrm{GL}(q, \mathbf{C})$.

REMARK. Let $z \in \widetilde{T}_q$. Let $a = {z^*}^{-1} z$. Then

$$\det a = \frac{\det z}{\overline{\det z}} = |\det z|^{-2} (\det z)^2 .$$

So $2 \arg\det z$ is a determination of $\arg(\det a)$. If z and z' are two matrices in $\widetilde{T}_q$ with the same angular matrix, then $\arg\det z$ and $\arg\det z'$ differ by an integral multiple of π. So the new invariant needed to characterize the orbits under $\mathrm{GL}(q, \mathbf{C})$ has to be regarded as a $\mathbf{Z}$-valued function. In this sense, it is a generalization of the signature.

4. The triple ratio on S

We return to the notation introduced in Sections 1 and 2.

For $z_1, z_2, z_3 \in \mathrm{Mat}(p \times q, \mathbf{C})$ define, whenever it makes sense, the element $T(z_1, z_2, z_3) \in \mathrm{GL}(q, \mathbf{C})$ by the following formula

$$\begin{aligned} T(z_1, z_2, z_3) &= k(z_1, z_2)\, k(z_3, z_2)^{-1} k(z_3, z_1) \\ &= (\mathbf{1}_q - z_2^* z_1)^{-1}(\mathbf{1}_q - z_2^* z_3)(\mathbf{1}_q - z_1^* z_3)^{-1} . \end{aligned} \tag{24}$$

It satisfies the following transformation law

$$T(g(z_1), g(z_2), g(z_3)) = j(g, z_1)\, T(z_1, z_2, z_3) j(g, z_1)^* \tag{25}$$

for $g \in G$. In particular, we see that $T(\sigma_1, \sigma_2, \sigma_3)$ is well defined on $S^3_{\top}$ and that the $\mathrm{GL}(q, \mathbf{C})$-orbit of $T(\sigma_1, \sigma_2, \sigma_3)$ is constant along any G-orbit in $S^3_{\top}$.

LEMMA 4.1. *Let* $\sigma = \begin{pmatrix} \sigma_p \\ \sigma' \end{pmatrix} \in S$, *tranverse to* ie *and* $-ie$. *Then*

$$T(ie, -ie, \sigma) = \frac{1}{2i}(i\mathbf{1}_q + \sigma_q)(\mathbf{1}_q + i\sigma_q)^{-1} . \tag{26}$$

Proof. This is an easy computation.

PROPOSITION 4.2. *Let* $(\sigma_1, \sigma_2, \sigma_3) \in S^3_{\top}$. *Then*

$$2i\ T(\sigma_1, \sigma_2, \sigma_3) \in \widetilde{T}_q^{(r)} .$$

Proof. Let us first assume $\sigma_1 = ie, \sigma_2 = -ie, \sigma_3 = \sigma$. Except for the factor $\frac{1}{2i}$, a comparison with (9) shows that $T(ie, -ie, \sigma)$ is the first term of the Cayley transform of σ. More precisely, let $c(\sigma) = \xi = \begin{pmatrix} \xi_q \\ \xi' \end{pmatrix}$. Then we may rewrite (26) as

$$T(ie, -ie, \sigma) = \frac{1}{2i}\xi_q .$$

Now ξ belongs to ${}^cS$, and hence $\frac{1}{2i}(\xi_q - \xi_q^*) = \xi'^* \xi'$. But $\mathrm{rank}(\xi') \leq r$, so $\mathrm{rank}(\xi'^* \xi') \leq r$ and hence ξ_q belongs to $\widetilde{T}_q^{(r)}$. Now the transformation law (25) for the triple ratio implies that for *any* $(\sigma_1, \sigma_2, \sigma_3) \in S^3_{\top}$, $2i\, T(\sigma_1, \sigma_2, \sigma_3)$ belongs to $\widetilde{T}_q^{(r)}$. $\square$

THEOREM 4.3. *Let* $(\sigma_1,\sigma_2,\sigma_3)$ *and* (τ_1,τ_2,τ_3) *belong to* $S^3_{\top}$. *They belong to the same* G*-orbit if and only if* $T(\sigma_1,\sigma_2,\sigma_3)$ *and* $T(\tau_1,\tau_2,\tau_3)$ *belong to the same* $\mathrm{GL}(q,\mathbf{C})$*-orbit.*

Proof. One way is obvious from the transformation law (25) for the triple ratio. For the converse, we assume (as we may) that $\sigma_1=\tau_1=ie$ and $\sigma_2=\tau_2=-ie$, and set for simplicity $\sigma=\sigma_3$ and $\tau=\tau_3$. Then the assumption implies that $(i\mathbf{1}_q+\sigma_q)(\mathbf{1}_q-i\sigma_q)^{-1}$ and $(i\mathbf{1}_q+\tau_q)(\mathbf{1}_q-i\tau_q)^{-1}$ are in the same $\mathrm{GL}(q,\mathbf{C})$-orbit. By Lemma 2.3, $c(\sigma)$ and $c(\tau)$ are in the same ${}^cL$-orbit. So σ and τ are in the same L-orbit. $\square$

Now to give a description of the invariant in terms of Theorem 3.13, we need to define the analog of the function $\arg\det$. For $z_1\in D$ and $z_2\in\overline{D}$, the function $k(z_1,z_2)=(\mathbf{1}_q-z_2^*z_1)^{-1}$ is well defined and belongs to $\mathrm{GL}(q,\mathbf{C})$. So we can extend the definition of T to the set

$$\widetilde{D}_{\top}=\{(z_1,z_2,z_3)\mid z_i\in D\cup S,\ 1\le i\le 3,\ z_1\top' z_2,\ z_2\top' z_3,\ z_3\top' z_1\},$$

where by definition $z\top' w$ is satisfied if z or w belongs to D, and reduces to the condition $z\top w$ if both z and w belong to S. As $\widetilde{D}_{\top}$ is stable by $(z_1,z_2,z_3)\mapsto(tz_1,tz_2,tz_3)$ for $0\le t\le 1$, this is a simply connected set. For $z_1\in D$, $\det T(z_1,z_1,z_1)$ is a positive real number. So there is a well defined continuous determination of the argument of $\det(T(z_1,z_2,z_3))$ on $\widetilde{D}_{\top}$ such that it takes the value 0 whenever $z_1=z_2=z_3\in D$. Denote this determination by $\arg\det T(z_1,z_2,z_3)$. It is clearly invariant under the G-action, and so it defines an invariant for the G-orbits.

On the other hand, let

$$S(z_1,z_2,z_3)=T(z_1,z_2,z_3)^{*^{-1}}T(z_1,z_2,z_3)$$

be the angular matrix associated to $T(z_1,z_2,z_3)$.

THEOREM 4.4. *Let* $(\sigma_1,\sigma_2,\sigma_3)$ *and* (τ_1,τ_2,τ_3) *belong to* $S^3_{\top}$. *They belong to the same* G*-orbit if and only if* $S(\sigma_1,\sigma_2,\sigma_3)$ *and* $S(\tau_1,\tau_2,\tau_3)$ *are conjugate under* $\mathrm{GL}(q,\mathbf{C})$ *and* $\arg\det T(\sigma_1,\sigma_2,\sigma_3)=\arg\det T(\tau_1,\tau_2,\tau_3)$.

Proof. This is a direct consequence of Theorem 4.3 and Theorem 3.13.

REMARK 1. Let us consider the case where $q=1$. The Stiefel manifold is the unit sphere $\mathbf{S}^{2p-1}$ in $\mathbf{C}^p$. The transversality condition $\sigma\top\tau$ just means $\sigma\neq\tau$, as is easily seen from the Cauchy-Schwarz inequality. The triple ratio is the complex number

$$T(\sigma_1, \sigma_2, \sigma_3) = (1 - \sigma_2^* \sigma_1)^{-1}(1 - \sigma_2^* \sigma_3)(1 - \sigma_1^* \sigma_3)^{-1} .$$

The group $\mathrm{GL}(q, \mathbf{C}) \simeq \mathbf{C}^*$ acts on the upper halfplane by $(\lambda, z) \mapsto |\lambda|^2 z$ and so the orbits are described by the argument of the complex number z. So the characteristic invariant in this case is just

$$\arg\left((1 - \sigma_2^* \sigma_1)^{-1}(1 - \sigma_2^* \sigma_3)(1 - \sigma_1^* \sigma_3)^{-1}\right) .$$

It is equivalent to the invariant θ considered in [KR]. This invariant, almost in our terms, was known to E. Cartan (see [Ca]).

REMARK 2. Let us consider the case where $p = q$. Then the Stiefel manifold is $\mathrm{U}(q)$, and the content of Proposition 4.2 is that for $(\sigma_1, \sigma_2, \sigma_3) \in S^3_{\top}$

$$T(\sigma_1, \sigma_2, \sigma_3) = (1 - \sigma_2^* \sigma_1)^{-1}(1 - \sigma_2^* \sigma_3)(1 - \sigma_1^* \sigma_3)^{-1}$$

is an invertible skew-Hermitian matrix. The orbits of $\mathrm{GL}(q, \mathbf{C})$ in its action on nondegenerate Hermitian forms are characterized by the signature. So the characteristic invariant as described in Theorem 4.3 in this case reduces to $\operatorname{sgn} iT(\sigma_1, \sigma_2, \sigma_3)$. As concerns Theorem 4.4, notice that the invariant S is trivial (equal to $-\mathbf{1}_q$), so one is only concerned with the invariant $\arg\det T$. The bounded domain D is of tube type and the description of the invariant through the function $\arg\det$ coincides with the approach of this problem in [CØ], where the invariant was introduced under the name of *generalized Maslov index*.

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