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A TRIPLE RATIO ON THE UNITARY STIEFEL MANIFOLD

by Jean-Louis CLERC

ABSTRACT. For the unitary Stiefel manifold S realized as the Shilov boundary of the unit ball D in $\text{Mat}(p \times q, \mathbf{C})$, we construct characteristic invariants for the (generic) orbits of the conformal group $\mathbf{PSU}(p, q)$ in $S \times S \times S$. The construction uses the automorphy kernel of the bounded symmetric domain.

INTRODUCTION

Let $D = G/K$ be a bounded symmetric domain in a complex vector space \mathbf{C}^N , and let S be its Shilov boundary. The action of G extends to S and this action is transitive on S . It is generally referred to in the literature as the *conformal action* of G on S . One can show that the action is almost 2-transitive in the sense that G has a dense open orbit in $S \times S$. Hence it is a natural question to look for the G -orbits in $S \times S \times S$ and for characteristic invariants of this action. If D happens to be of tube type (in which case $\dim_{\mathbf{R}} S = \dim_{\mathbf{C}} D$), this question was solved in [CØ]. There are a finite number of open orbits in $S \times S \times S$, and the (generalized) *Maslov index* we constructed is a characteristic invariant for the G -action. In the case of the unit ball in \mathbf{C}^2 , the Shilov boundary coincides with the topological boundary, namely the unit sphere $S = \mathbf{S}^3$. In [Ca], E. Cartan constructed a (real-valued) invariant for triples on S (he called S the “hypersphere”). Independently (and more than 50 years later) Korányi and Reimann studied the case of the unit ball in \mathbf{C}^n (see [KR]). Through the Cayley transform, the problem is changed into an equivalent problem for the Heisenberg group \mathbf{H}_n under the action of its conformal group $G = \mathbf{PSU}(n + 1, 1)$. For this situation, they studied a complex cross ratio on \mathbf{H}_n , from which they were able (in a rather indirect way) to construct a (real-valued) invariant for triples, which characterizes the G -orbits of triples in \mathbf{H}_n . Here we solve the problem for the case where D

is the unit ball in the matrix space $\text{Mat}(p \times q, \mathbf{C})$, S is the unitary Stiefel manifold $\mathbf{S}_{p,q}$ and $G = \mathbf{PSU}(p, q)$. The invariant we construct for triples is of matrix-valued nature (it is a conjugacy class) and we give two versions of it (see Theorems 4.3 and 4.4). The basic strategy is to approach the Shilov boundary from inside. The (matrix-valued) *automorphy kernel* for the domain D is used to build a kernel for triples of points inside D which transforms nicely under the action of G . It remains to look carefully at the boundary behaviour of the kernel when the points approach the Shilov boundary S . This is only possible for triples satisfying a generic condition called *transversality* (see Proposition 2.1 for a definition). The *Cayley transform* plays an important role in the proofs. Finally the problem is reduced to a *linear* problem, which is related to the description of some orbits for the action $(g, X) \mapsto gXg^*$ of GL_q on $\text{Mat}(q \times q, \mathbf{C})$ (see Theorem 3.9).

For general references on bounded symmetric domains and their geometric properties, see [S], and Part III in [Fal]. For explicit calculations related to our example, see [P] and [H].

1. GEOMETRIC SETTING

Let p, q be two integers with $1 \leq q \leq p$, and let

$$(1) \quad D = \{z \in \text{Mat}(p \times q, \mathbf{C}) \mid \mathbf{1}_q - z^*z \gg 0\}.$$

Let $G = \text{SU}(p, q) \subset \text{GL}(p + q, \mathbf{C})$. An element $g \in \text{GL}(p + q, \mathbf{C})$ will often be written as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where

$$a \in \text{Mat}(p \times p, \mathbf{C}), \quad b \in \text{Mat}(p \times q, \mathbf{C}), \quad c \in \text{Mat}(q \times p, \mathbf{C}), \quad d \in \text{Mat}(q \times q, \mathbf{C}).$$

In this notation, the conditions for g to belong to $\text{U}(p, q, \mathbf{C})$ can be written as

$$(2) \quad \begin{aligned} a^*a - c^*c &= \mathbf{1}_p \\ b^*a - d^*c &= 0 \\ d^*d - b^*b &= \mathbf{1}_q. \end{aligned}$$

Define an action of the group $\text{GL}(p + q, \mathbf{C})$ on $\text{Mat}(p \times q, \mathbf{C})$ by

$$(3) \quad g(z) = (az + b)(cz + d)^{-1}.$$

The action is not everywhere defined, but it is certainly defined if $g \in G$ and $z \in D$. It defines an action of G on D , and G (or rather $\text{PSU}(p, q)$) is the neutral component of the group of all biholomorphic transformations of D .

The stabilizer of the base point $0 \in D$ is the maximal compact subgroup $K = S(\text{U}(p) \times \text{U}(q))$. Its complexification is the complex group $K^{\mathbf{C}} = S(\text{GL}(p, \mathbf{C}) \times \text{GL}(q, \mathbf{C}))$. We also define the following subgroups

$$P^+ = \left\{ \begin{pmatrix} \mathbf{1}_p & z \\ 0 & \mathbf{1}_q \end{pmatrix}, z \in \text{Mat}(p \times q, \mathbf{C}) \right\}$$

$$P^- = \left\{ \begin{pmatrix} \mathbf{1}_p & 0 \\ w & \mathbf{1}_q \end{pmatrix}, w \in \text{Mat}(q \times p, \mathbf{C}) \right\}.$$

The corresponding *Harish Chandra decomposition* is the following identity

$$(4) \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \mathbf{1}_p & bd^{-1} \\ 0 & \mathbf{1}_q \end{pmatrix} \begin{pmatrix} a - bd^{-1}c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} \mathbf{1}_p & 0 \\ d^{-1}c & \mathbf{1}_q \end{pmatrix}$$

valid for $g \in \text{GL}(p + q, \mathbf{C})$ if d is invertible.

The *automorphy kernel* $k(z, w)$ is defined for $z, w \in \text{Mat}(p \times q, \mathbf{C})$ wherever it makes sense by the formula

$$(5) \quad k(z, w) = (\mathbf{1}_q - w^*z)^{-1}.$$

In particular it is always well defined for $z, w \in D$ and has values in $\text{GL}(q, \mathbf{C})$. It has the following law of transformation for $g \in G$

$$(6) \quad k(g(z), g(w)) = j(g, z)k(z, w)j(g, w)^*,$$

where

$$(7) \quad j(g, z) = cz + d.$$

The *Shilov boundary* of D is the unitary Stiefel manifold S defined by

$$(8) \quad S = \{ \sigma \in \text{Mat}(p \times q, \mathbf{C}) \mid \sigma^* \sigma = \mathbf{1}_q \}.$$

The action of G extends to S , and it is clearly transitive on S . In fact the action of K is already transitive.

To go further, we need to make a specific choice of a base point in S . For this we first systematically write elements in $\text{Mat}(p \times q, \mathbf{C})$ as

$$z = \begin{pmatrix} z_q \\ z' \end{pmatrix}$$

where $z_q \in \text{Mat}(q \times q, \mathbf{C})$ and $z' \in \text{Mat}((p - q) \times q, \mathbf{C})$. With this convention, let $ie = \begin{pmatrix} i\mathbf{1}_q \\ 0 \end{pmatrix}$ be the base point in S . Associated to this choice is the Cayley transform c , given by

$$z = \begin{pmatrix} z_q \\ z' \end{pmatrix} \mapsto c(z) = \begin{pmatrix} w_q \\ w' \end{pmatrix}$$

with

$$(9) \quad \begin{aligned} w_q &= (z_q + i\mathbf{1}_q)(iz_q + \mathbf{1}_q)^{-1} \\ w' &= -z'(iz_q + \mathbf{1}_q)^{-1}. \end{aligned}$$

The inverse of the Cayley transform is the map which to $\begin{pmatrix} w_q \\ w' \end{pmatrix}$ associates the matrix $\begin{pmatrix} z_q \\ z' \end{pmatrix}$ given by

$$(10) \quad \begin{aligned} z_q &= (iw_q - \mathbf{1}_q)^{-1}(i\mathbf{1}_q - w_q) \\ z' &= 2w'(iw_q - \mathbf{1}_q)^{-1}. \end{aligned}$$

The Cayley transform is a rational map, well defined on D . The image of D is the *Siegel domain of type II* defined by

$$(11) \quad {}^cD = \left\{ \begin{pmatrix} w_q \\ w' \end{pmatrix}, \frac{1}{2i}(w_q - w_q^*) - w'^*w' \gg 0 \right\}$$

and the image of the Shilov boundary (more exactly the part of the Shilov boundary where the Cayley transform is defined) is

$$(12) \quad {}^cS = \left\{ \begin{pmatrix} w_q \\ w' \end{pmatrix}, \frac{1}{2i}(w_q - w_q^*) = w'^*w' \right\}.$$

To the data

$$w_0 \in \text{Mat}((p - q) \times q, \mathbf{C})$$

$$h \in \text{GL}(q, \mathbf{C}), u \in \text{U}(p - q, \mathbf{C}), \text{ such that } \det h = (\det u)^{-1}$$

$$s \in \text{Herm}(q, \mathbf{C})$$

we associate the transform

$$(13) \quad \begin{aligned} w_q &\mapsto h^*w_qh + s + 2iw_0^*uw'h + iw_0^*w_0 \\ w' &\mapsto uw'h + w_0. \end{aligned}$$

Any such transform maps cD in a one-to-one fashion into itself. These transforms form a group and it is exactly the group of affine holomorphic transforms of the domain cD .

Let B be the stabilizer of the point ie in G . The conjugate group under the Cayley transform is ${}^cB = c \circ B \circ c^{-1}$ and it turns out to be exactly the group of affine transforms of cD we just described. Observe that the group cB is transitive on cD and on cS .

2. ACTION OF G ON $S \times S$ AND $S \times S \times S$

We now study the action of G on pairs of points of S . The main notion to be introduced is *transversality*, a notion that could be defined for any bounded symmetric domain. We give several equivalent definitions for our case.

PROPOSITION 2.1. *Let σ and ξ be two elements of S . Then the following are equivalent:*

- (i) $\det(\mathbf{1}_q - \xi^* \sigma) \neq 0$;
- (ii) $\xi - \sigma$ injective;
- (iii) $\det(\mathbf{1}_p - \xi \sigma^*) \neq 0$.

If one of these equivalent conditions is satisfied, then σ and ξ are said to be transverse.

Proof. Assume (i). As $\mathbf{1}_q = \xi^* \xi$, this condition amounts to $\det(\xi^*(\xi - \sigma)) \neq 0$, which in particular shows that $\xi - \sigma$ is injective. Conversely, assume $\xi - \sigma$ is injective and let $v \in \mathbf{C}^q$ be such that $v = \xi^* \sigma v$. Now

$$\|v\| = \|\xi^* \sigma v\| \leq \|\sigma v\| \leq \|v\|,$$

and hence $\|\xi^* \sigma v\| = \|\sigma v\|$, which is possible only if $\sigma v \in \text{Im } \xi$. So there exists $w \in \mathbf{C}^q$, such that $\sigma v = \xi w$. But taking the image of both sides by ξ^* yields $v = w$, and hence $\sigma v = \xi v$, so that $v = 0$. So $\mathbf{1}_q - \xi^* \sigma$ is injective and hence (ii) \implies (i). Under the same assumption (ii), let us prove that $\xi \sigma^*$ cannot have 1 as an eigenvalue. Suppose $v \in \mathbf{C}^p$ is such that $\xi \sigma^* v = v$. As ξ is a partial isometry, this forces $\|\sigma^* v\| = \|v\|$, and hence v belongs to the image of the map σ , so there exists $w \in \mathbf{C}^q$ such that $v = \sigma w$. But then we also have $v = \xi \sigma^* \sigma w = \xi w$ and hence $(\sigma - \xi)w = 0$ which forces $w = 0$. Hence (iii) follows from (ii). Finally assume (iii). Then as σ is injective, $(\mathbf{1}_p - \xi \sigma^*) \circ \sigma = \sigma - \xi$ is also injective. Hence (iii) \implies (ii). \square

We will use the notation $\sigma \top \xi$ to denote transversality. It is a symmetric condition. It is invariant under the action of G , as can easily be concluded from (6). For $\sigma \in S$, let

$$S_{\top}^{\sigma} = \{\xi \mid \sigma \top \xi\}.$$

Observe that the set S_{\top}^{ie} is exactly the subset in S where the Cayley transform is defined.

Let

$$(14) \quad S_{\top}^2 = \{(\sigma, \xi) \in S \times S \mid \sigma \top \xi\}.$$

As base point in S_{\top}^2 we choose $(ie, -ie)$. Observe that $c(-ie) = 0$.

THEOREM 2.2. *The group G acts transitively on S_{\top}^2 .*

Proof. Let $(\sigma, \xi) \in S_{\top}^2$ and let us show that there exists an element of G which maps (σ, ξ) to $(ie, -ie)$. As G is transitive on S , we may assume that $\sigma = ie$. Then the transversality condition shows that ξ belongs to the domain of the Cayley transform. The element $c(\xi)$ belongs to cS , and we have already noticed that cB is transitive on cS . Hence $c(\xi)$ can be mapped to $0 = c(-ie)$. Taking the image under the inverse Cayley transform gives the result. \square

Denote by L the stabilizer of the base point $(ie, -ie)$ in B . Under a Cayley transform, the group ${}^cL = c \circ L \circ c^{-1}$ is the stabilizer in cB of the element 0. Hence it is the subgroup of linear transformations given by

$$\begin{aligned} w_q &\longmapsto h^* w_q h \\ w' &\longmapsto u w h \end{aligned}$$

where $h \in \text{GL}(q, \mathbf{C})$, $u \in \text{U}(p - q)$ and $\det h = (\det u)^{-1}$.

LEMMA 2.3. *Let $\begin{pmatrix} w_q \\ w' \end{pmatrix}, \begin{pmatrix} v_q \\ v' \end{pmatrix} \in {}^cS$. Then they belong to the same orbit under the action of cL if and only if w_q and v_q belong to the same orbit under the action of $\text{GL}(q, \mathbf{C})$.*

Proof. One implication being trivial, we only have to prove the other one. So assume there exists $h \in \text{GL}(q, \mathbf{C})$ such that $v_q = h^* w_q h$. Let μ be a complex number such that $\mu^{p-q} = \det h$ and let $u = \mu^{-1} \mathbf{1}_{p-q}$. Clearly $(\det u)^{-1} = \det h$. Using the action of (h, u) we may assume that $v_q = w_q$. Let $s_q = \frac{1}{2i}(w_q - w_q^*)$. This is an Hermitian matrix and as w_q and v_q belong to cS , we get

$$s_q = w'^* w' = v'^* v'.$$

Looking to the columns of w' (or v'), we may think of w' as a family of q vectors in \mathbf{C}^{p-q} . Then the matrix s_q is the Gram matrix of these vectors. But two sets of vectors in \mathbf{C}^{p-q} are conjugate under the action of the unitary group $\text{U}(p - q)$ if and only if they have the same Gram matrix. Hence there exists $u \in \text{U}(p - q)$ such that $v' = u w'$. Let λ be a complex number such that $\lambda^q = \det u$. Then using the action of $(\lambda^{-1} \mathbf{1}_q, u)$, we get the result. \square

Let us denote by H_q the real vector space of $q \times q$ Hermitian matrices, and let Ω_q be the subset of all positive-definite matrices. For any integer r such that $0 \leq r \leq q$ let $\Omega_q^{(r)}$ be the set of all positive semi-definite $q \times q$ Hermitian matrices of rank less than r . For $r < q$, the set $\Omega_q^{(r)}$ is contained in the boundary of Ω_q , whereas for $r = q$, $\Omega_q^{(q)} = \overline{\Omega}_q$.

Let

$$T_q^{(r)} = \{x + iy \mid x \in H_q, y \in \Omega_q^{(r)}\}.$$

The group $GL(q, \mathbf{C})$ acts on $T_q^{(r)}$ by the action $(h, w) \mapsto hwh^*$.

Finally let

$$\tilde{T}_q^{(r)} = \{z \in T_q^{(r)} \mid z \text{ invertible}\}.$$

Clearly the action of $GL(q, \mathbf{C})$ preserves $\tilde{T}_q^{(r)}$.

Let $\begin{pmatrix} w_q \\ w' \end{pmatrix}$ be in cS . Then $w_q = x_q + iw'^*w'$, with $x_q \in H_q$. Let

$$r = \inf(q, p - q).$$

The rank of the matrix w'^*w' is at most r . Hence w_q belongs to $T_q^{(r)}$. Conversely, it is easily seen that any positive semi-definite Hermitian matrix of rank at most r can be written as w'^*w' for some $w' \in \text{Mat}((p - q) \times q, \mathbf{C})$.

Let

$$(15) \quad S_{\top}^3 = \{(\sigma_1, \sigma_2, \sigma_3) \in S \times S \times S \mid \sigma_1 \top \sigma_2, \sigma_2 \top \sigma_3, \sigma_3 \top \sigma_1\}.$$

THEOREM 2.4. *The G -orbits in S_{\top}^3 are in one-to-one correspondance with the orbits of $GL(q, \mathbf{C})$ in $\tilde{T}_q^{(r)}$.*

Proof. From Theorem 2.2 we already know that any orbit contains an element of the form $(ie, -ie, \sigma)$ with $\sigma \in S$. Now use the Cayley transform. The element $w = c(\sigma)$ is in cS , and the transversality condition is equivalent to the condition $\det(w_q) \neq 0$. In other words, $w_q \in \tilde{T}_q^{(r)}$. The result now follows from Lemma 2.3. \square

3. ORBITS FOR THE GL_q -ACTION ON \tilde{T}_q

Any $z \in \text{Mat}(q \times q, \mathbf{C})$ can be written in a unique way as $z = x + iy$, with $x, y \in H_q$. We will be concerned with the set \tilde{T}_q defined by

$$(16) \quad \tilde{T}_q = \{z \in \text{Mat}(q \times q, \mathbf{C}) \mid z = x + iy, x \in H_q, y \in \overline{\Omega}_q, \det z \neq 0\}.$$

Its interior is the classical *tube domain* over the cone Ω_q , namely

$$T_q = \{z \in \text{Mat}(q \times q, \mathbf{C}) \mid z = x + iy, y \in \Omega_q\}.$$

Let $G = GL(q, \mathbf{C})$ act on $\text{Mat}(q \times q, \mathbf{C})$ by

$$(17) \quad (g, z) \longmapsto gzg^*.$$

The spaces $H_q, \Omega_q, \overline{\Omega}_q$ are stable under this action, and hence \tilde{T}_q and T_q are invariant subsets under this action. We investigate the orbits and describe a full set of invariants for this action.

There is a natural invariant associated to a $GL(q, \mathbf{C})$ -orbit. To any $z \in \tilde{T}_q$, we associate its *angular matrix* defined by

$$(18) \quad a = a(z) = z^{*-1}z.$$

Then the matrix associated to gzg^* is $g^{*-1}ag^*$, so that the angular matrix $a(z)$ belongs to the same conjugacy class when z runs through a $GL(q, \mathbf{C})$ -orbit. As we shall see (Theorem 3.3 and Theorem 3.13), this invariant is close to characterizing the orbits.

Let us first prove some elementary properties of the angular matrix.

PROPOSITION 3.1. *Let $z = x + iy \in \tilde{T}_q$, and let $a = z^{*-1}z$ be its angular matrix. Then*

- (i) $\text{Sp}(a) \subset U_1 = \{\mu \in \mathbf{C}, |\mu| = 1\}$;
- (ii) if $1 \in \text{Sp}(a)$, then y is degenerate and

$$\{v \in \mathbf{C}^q \mid av = v\} = \{v \in \mathbf{C}^q \mid yv = 0\}.$$

Proof. Let μ be an eigenvalue of a , and let $v \neq 0$ be an eigenvector for the eigenvalue μ . Then $zv = \mu z^*v$, and hence

$$(zv, v) = \mu(z^*v, v) = \mu(v, zv) = \mu \overline{(zv, v)}.$$

If $(zv, v) \neq 0$, then $|\mu| = 1$. So we now assume $(zv, v) = 0$. This amounts to $(xv, v) + i(yv, v) = 0$, so that in particular $(yv, v) = 0$. Now recall that y is positive semi-definite. So the condition $(yv, v) = 0$ implies that $yv = 0$. From this it follows that $zv = xv = z^*v$, and as z is assumed to be invertible, this implies $\mu = 1$. This shows (i) and part of (ii). Conversely, the condition $yv = 0$ implies trivially $av = v$. \square

In particular, we may consider the polynomial $d(\mu) = \det(z - \mu z^*)$. The roots of d are the eigenvalues of the angular matrix. The set of these roots, counted with their multiplicities, will be called the *angular spectrum* of z .

We first consider the case of T_q . So let $z = x + iy \in T_q$. Then as y is positive-definite, we may define its square root $y^{1/2}$ as the unique positive-definite Hermitian matrix whose square is y . Then we may write

$$x + iy = y^{\frac{1}{2}}(y^{-\frac{1}{2}}xy^{-\frac{1}{2}} + i\mathbf{1}_q)y^{\frac{1}{2}}.$$

This shows that any $GL(q, \mathbf{C})$ -orbit contains some element of the form $x + i\mathbf{1}_q$, where $x \in H_q$. But by the classical diagonalization theorem for Hermitian forms, there exists an orthonormal basis in which the Hermitian form associated to x is diagonal. In other words, there exists a unitary matrix u and real numbers $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_q$ such that

$$uxu^* = \Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_q \end{pmatrix}.$$

Moreover, if Λ and Λ' are two such diagonal matrices, then $\Lambda + i\mathbf{1}_q$ and $\Lambda' + i\mathbf{1}_q$ are not conjugate under the action of $GL(q, \mathbf{C})$ unless $\Lambda = \Lambda'$. Hence we have shown the following result, which of course is the well-known fact that there is a simultaneous diagonalization for two Hermitian forms if one of them is positive-definite.

THEOREM 3.2. *The set of matrices of the form*

$$(19) \quad \Lambda = \begin{pmatrix} \lambda_1 + i & & & \\ & \lambda_2 + i & & \\ & & \ddots & \\ & & & \lambda_q + i \end{pmatrix}$$

with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_q$ is a full set of representatives of the $GL(q, \mathbf{C})$ -orbits in T_q .

The angular matrix associated to Λ is

$$(20) \quad \begin{pmatrix} \frac{\lambda_1+i}{\lambda_1-i} & & & \\ & \frac{\lambda_2+i}{\lambda_2-i} & & \\ & & \ddots & \\ & & & \frac{\lambda_q+i}{\lambda_q-i} \end{pmatrix}.$$

Now we study the action of G_r in H_n . If $x \in H_n$, let us write

$$x = \begin{pmatrix} \alpha & b \\ b^* & \gamma \end{pmatrix}$$

where $\alpha \in H_r$, $b \in \text{Mat}(r \times s, \mathbf{C})$ and $\gamma \in H_s$. If $g = \begin{pmatrix} u & v \\ 0 & h \end{pmatrix} \in G_r$, then $gxg^* = \begin{pmatrix} \alpha' & b' \\ b'^* & \gamma' \end{pmatrix}$, with

$$\alpha' = u\alpha u^* + ubv^* + vb^*u^* + v\gamma v^*$$

$$b' = ubh^* + v\gamma h^*$$

$$\gamma' = h\gamma h^*.$$

LEMMA 3.6. Let $x = \begin{pmatrix} \alpha & b \\ b^* & \gamma \end{pmatrix} \in H_n$, with $\alpha \in H_r$, $b \in \text{Mat}(r \times s, \mathbf{C})$ and $\gamma \in H_s$. Assume $\det \gamma \neq 0$. Then the orbit of x under G_r contains a matrix of the form $\begin{pmatrix} \alpha' & 0 \\ 0 & \gamma \end{pmatrix}$ with $\alpha' \in H_r$.

Proof. This is a consequence of the previous formula with $u = \mathbf{1}_r$, $v = -b\gamma^{-1}$ and $h = \mathbf{1}_s$.

LEMMA 3.7. Let $x = \begin{pmatrix} \alpha & b \\ b^* & 0 \end{pmatrix} \in H_n$, with $\text{rank } b = s$ (so in particular $r \geq s$). Then the orbit of x under G_r contains an element of the form

$$\begin{pmatrix} \beta & 0 & 0 \\ 0 & 0 & \mathbf{1}_s \\ 0 & \mathbf{1}_s & 0 \end{pmatrix}$$

with $\beta \in H_{r-s}$.

Proof. Consider the subgroup $\left\{ \begin{pmatrix} u & 0 \\ 0 & h \end{pmatrix}, u \in U(r), h \in GL_s(\mathbf{C}) \right\}$. It acts on the component b by $b' = ubh^*$. As $\text{rank}(b) = s$, we may think of b as a set of s independent vectors in \mathbf{C}^r . By the Gram-Schmidt process, it is possible to find $h \in GL_s(\mathbf{C})$ such that bh^* is a s -orthonormal frame in \mathbf{C}^r . But now two such frames are conjugate by the (left) action of $U(r)$. Hence there exists $u \in U(r)$ such that

$$ubh^* = \begin{pmatrix} 0 \\ \mathbf{1}_s \end{pmatrix}.$$

The matrix x we started with is conjugate under G_r to a matrix of the form

$$\begin{pmatrix} \alpha' & c & 0 \\ c^* & \beta & \mathbf{1}_s \\ 0 & \mathbf{1}_s & 0 \end{pmatrix}$$

where $\alpha' \in H_{r-s}$, $\beta \in H_s$ and $c \in \text{Mat}((r-s) \times s, \mathbf{C})$. Now we use the action of the element

$$g = \begin{pmatrix} \mathbf{1}_{r-s} & 0 & -c \\ 0 & \mathbf{1}_s & -\frac{\beta}{2} \\ 0 & 0 & \mathbf{1}_s \end{pmatrix} \in G_r$$

to get the result. \square

We are now ready to start the proof of Theorem 3.4.

STEP 1. Let $z = x + iy \in \tilde{T}_q$. As y is positive semidefinite, there exists an element $g \in \text{GL}(q, \mathbf{C})$ such that

$$gyg^* = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix},$$

with r diagonal entries equal to 1, and s diagonal entries equal to 0, r and s being nonnegative integers satisfying $r + s = q$. In other terms, any $\text{GL}(q, \mathbf{C})$ -orbit in \tilde{T}_q contains an element of the form

$$\begin{pmatrix} \alpha + i\mathbf{1}_r & b \\ b^* & \gamma \end{pmatrix}$$

with $\alpha \in H_r, \gamma \in H_s, b \in \text{Mat}(r \times s, \mathbf{C})$.

STEP 2. Now assume x is of the form

$$x = \begin{pmatrix} \alpha + i\mathbf{1}_r & b \\ b^* & \gamma \end{pmatrix}.$$

Consider γ . It is an Hermitian matrix of size s , and under the action of $\text{GL}(s, \mathbf{C})$ it can be transformed to

$$\begin{pmatrix} \mathbf{0}_{n_2} & 0 & 0 \\ 0 & \mathbf{1}_{n_3} & 0 \\ 0 & 0 & -\mathbf{1}_{n_4} \end{pmatrix}$$

where $n_2 + n_3 + n_4 = s$. Hence x is conjugate under the action of G_r to an element of the form

$$\begin{pmatrix} \alpha & b' & c' \\ b'^* & 0 & 0 \\ c'^* & 0 & \Upsilon \end{pmatrix}$$

where $\alpha \in H_r$, $b' \in \text{Mat}(r \times n_2, \mathbf{C})$, $c' \in \text{Mat}(r \times (n_3 + n_4), \mathbf{C})$ and

$$\Upsilon = \begin{pmatrix} \mathbf{1}_{n_3} & 0 \\ 0 & -\mathbf{1}_{n_4} \end{pmatrix}.$$

Using Lemma 3.6, we see that x is conjugate under the action of G_s to an element of the form

$$\begin{pmatrix} \alpha'' & b'' & 0 \\ b''^* & 0 & 0 \\ 0 & 0 & \Upsilon \end{pmatrix},$$

with $\alpha'' \in H_r$, $b'' \in \text{Mat}(r \times n_2, \mathbf{C})$.

STEP 3. Assume now that

$$x = \begin{pmatrix} \alpha & b & 0 \\ b^* & 0 & 0 \\ 0 & 0 & \Upsilon \end{pmatrix}$$

with $\alpha \in H_r$ and $b \in \text{Mat}(r \times n_2, \mathbf{C})$. Recall that

$$x + iy = \begin{pmatrix} \alpha + i\mathbf{1}_r & b & 0 \\ b^* & 0 & 0 \\ 0 & 0 & \Upsilon \end{pmatrix}$$

is assumed to be invertible. This shows that $\text{rank}(b) = n_2$. So we may apply Lemma 3.7 to see that x is conjugate under G_r to an element of the form

$$\begin{pmatrix} \beta & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1}_{n_2} & 0 \\ 0 & \mathbf{1}_{n_2} & 0 & 0 \\ 0 & 0 & 0 & \Upsilon \end{pmatrix}$$

with $\beta \in H_{r-n_2}$.

STEP 4. Set $n_1 = r - n_2$. The last step is just to put the element $\beta \in H_{n_1}$ in diagonal form under the action of $U(n_1)$. Up to minor rearrangements of the matrix, this shows that any $\text{GL}(q, \mathbf{C})$ -orbit in \tilde{T}_q contains an element of the form $\Lambda(\lambda_1, \dots, \lambda_{n_1}, n_2, n_3, n_4)$.

THEOREM 3.9. *Any $\text{GL}(q, \mathbf{C})$ -orbit in $\widetilde{T}_q^{(r)}$ contains a unique standard matrix $\Lambda((\lambda_1, \dots, \lambda_{n_1}, n_2, n_3, n_4))$ with $n_1 + n_2 \leq r$.*

We now want an analog of Theorem 3.3. As we have already noticed, the conjugacy class of the angular matrix does not determine the orbit of the matrix. We need a finer invariant, which we will construct now.

LEMMA 3.10. *The space \widetilde{T}_q is connected and simply connected.*

Proof. As T_q is connected and $T_q \subset \widetilde{T}_q \subset \overline{T}_q$, the space \widetilde{T}_q is connected. Take $i\mathbf{1}_q$ as base point in \widetilde{T}_q , and observe that for any $z \in \widetilde{T}_q$ and any $s > 0$, $z + is\mathbf{1}_q$ is in T_q . So if $(\gamma(t), t \in [0, 1])$ is a path in \widetilde{T}_q starting and ending at $i\mathbf{1}_q$ then we can deform it by homotopy to $\gamma_s(t) = \gamma(t) + is(s-1)\mathbf{1}_q$, which for $s > 0$ is a path inside T_q . But T_q as a tube-type domain is simply connected. \square

The function $z \mapsto \det(z)$ is a continuous function from \widetilde{T}_q into \mathbf{C}^* . From Lemma 3.10, there exists a unique continuous determination of the argument of $\det(z)$ denoted by $\arg \det: \widetilde{T}_q \rightarrow \mathbf{R}$ such that $\arg \det i\mathbf{1}_q = q\frac{\pi}{2}$. If $Y \in \Omega_q$, then $\arg \det iy = q\frac{\pi}{2}$. If $z \in \widetilde{T}_q$ and $g \in \text{GL}(q, \mathbf{C})$, then $\det gzg^* = |\det g|^2 \det z$, and $gi\mathbf{1}_qg^* = igg^* \in i\Omega_q$, so that

$$\arg \det gzg^* = \arg \det z.$$

This provides a new invariant for the action of $\text{GL}(q, \mathbf{C})$ on \widetilde{T}_q .

LEMMA 3.11. *Let $\Lambda = \Lambda(\lambda_1, \dots, \lambda_{n_1}, n_2, n_3, n_4)$. Then*

$$(23) \quad \arg \det \Lambda = \arg(\lambda_1 + i) + \dots + \arg(\lambda_{n_1} + i) + n_2\pi + n_4\pi$$

where \arg is used for the principal determination of the argument of a non-zero complex number.

Proof. We need to describe a continuous path from $i\mathbf{1}_q$ to Λ inside \widetilde{T}_q . For clarity of exposition, we describe successively the path for each diagonal block (either a one-dimensional or a two-dimensional submatrix) of Λ , and compute the contribution of each block to the function $\arg \det$.

For a block of the form $\lambda + i$, with $\lambda \in \mathbf{R}$ we use the path $t \mapsto t\lambda + i$, $0 \leq t \leq 1$, and so the contribution of this block is $\arg(\lambda + i)$.

For a block of the form $\begin{pmatrix} i & 1 \\ 1 & 0 \end{pmatrix}$, we use the path

$$t \mapsto \begin{pmatrix} i & t \\ t & i(1-t^2) \end{pmatrix}, \quad 0 \leq t \leq 1.$$

The corresponding determinant of this 2×2 -block is constant along the path and equal to -1 . Hence the contribution of this block is $2\frac{\pi}{2} = \pi$.

For a block of the form 1 , we use the path $t \mapsto e^{i\frac{\pi}{2}(1-t)}$, $0 \leq t \leq 1$, and we see that the corresponding contribution is 0 .

For a block of the form -1 , we use the path $t \mapsto e^{i\frac{\pi}{2}(1+t)}$, $0 \leq t \leq 1$, and we see that the corresponding contribution is π .

Putting together the contribution of the blocks, we get the result. \square

COROLLARY 3.12. *Let Λ and Λ' be two standard matrices. Assume that their angular matrices coincide and that $\arg \det \Lambda = \arg \det \Lambda'$. Then $\Lambda = \Lambda'$.*

Proof. In fact we noticed that the equality of angular matrices implies the equality of the parameters except for $n_3 = n'_3$ and $n_4 = n'_4$. But from (23), we see that the equality of the determination of the arguments of the determinants implies $n_4 = n'_4$ (and hence $n_3 = n'_3$). \square

Now we can state the conclusion of this section, which is a consequence of Theorem 3.4 and Corollary 3.12.

THEOREM 3.13. *Let $z, z' \in \tilde{T}_q$, and assume that the angular matrices of z and z' are conjugate, and that $\arg \det z = \arg \det z'$. Then z and z' belong to the same orbit under the action of $\mathrm{GL}(q, \mathbf{C})$.*

REMARK. Let $z \in \tilde{T}_q$. Let $a = z^{*-1}z$. Then

$$\det a = \frac{\det z}{\det z} = |\det z|^{-2}(\det z)^2.$$

So $2 \arg \det z$ is a determination of $\arg(\det a)$. If z and z' are two matrices in \tilde{T}_q with the same angular matrix, then $\arg \det z$ and $\arg \det z'$ differ by an integral multiple of π . So the new invariant needed to characterize the orbits under $\mathrm{GL}(q, \mathbf{C})$ has to be regarded as a \mathbf{Z} -valued function. In this sense, it is a generalization of the signature.

4. THE TRIPLE RATIO ON S

We return to the notation introduced in Sections 1 and 2.

For $z_1, z_2, z_3 \in \text{Mat}(p \times q, \mathbf{C})$ define, whenever it makes sense, the element $T(z_1, z_2, z_3) \in \text{GL}(q, \mathbf{C})$ by the following formula

$$(24) \quad \begin{aligned} T(z_1, z_2, z_3) &= k(z_1, z_2) k(z_3, z_2)^{-1} k(z_3, z_1) \\ &= (\mathbf{1}_q - z_2^* z_1)^{-1} (\mathbf{1}_q - z_2^* z_3) (\mathbf{1}_q - z_1^* z_3)^{-1}. \end{aligned}$$

It satisfies the following transformation law

$$(25) \quad T(g(z_1), g(z_2), g(z_3)) = j(g, z_1) T(z_1, z_2, z_3) j(g, z_1)^*$$

for $g \in G$. In particular, we see that $T(\sigma_1, \sigma_2, \sigma_3)$ is well defined on S_{\top}^3 and that the $\text{GL}(q, \mathbf{C})$ -orbit of $T(\sigma_1, \sigma_2, \sigma_3)$ is constant along any G -orbit in S_{\top}^3 .

LEMMA 4.1. Let $\sigma = \begin{pmatrix} \sigma_p \\ \sigma' \end{pmatrix} \in S$, tranverse to ie and $-ie$. Then

$$(26) \quad T(ie, -ie, \sigma) = \frac{1}{2i} (i\mathbf{1}_q + \sigma_q) (\mathbf{1}_q + i\sigma_q)^{-1}.$$

Proof. This is an easy computation.

PROPOSITION 4.2. Let $(\sigma_1, \sigma_2, \sigma_3) \in S_{\top}^3$. Then

$$2i T(\sigma_1, \sigma_2, \sigma_3) \in \tilde{T}_q^{(r)}.$$

Proof. Let us first assume $\sigma_1 = ie, \sigma_2 = -ie, \sigma_3 = \sigma$. Except for the factor $\frac{1}{2i}$, a comparison with (9) shows that $T(ie, -ie, \sigma)$ is the first term of the Cayley transform of σ . More precisely, let $c(\sigma) = \xi = \begin{pmatrix} \xi_q \\ \xi' \end{pmatrix}$. Then we may rewrite (26) as

$$T(ie, -ie, \sigma) = \frac{1}{2i} \xi_q.$$

Now ξ belongs to cS , and hence $\frac{1}{2i}(\xi_q - \xi_q^*) = \xi'^* \xi'$. But $\text{rank}(\xi') \leq r$, so $\text{rank}(\xi'^* \xi') \leq r$ and hence ξ_q belongs to $\tilde{T}_q^{(r)}$. Now the transformation law (25) for the triple ratio implies that for any $(\sigma_1, \sigma_2, \sigma_3) \in S_{\top}^3$, $2i T(\sigma_1, \sigma_2, \sigma_3)$ belongs to $\tilde{T}_q^{(r)}$. \square

THEOREM 4.3. *Let $(\sigma_1, \sigma_2, \sigma_3)$ and (τ_1, τ_2, τ_3) belong to S_{\top}^3 . They belong to the same G -orbit if and only if $T(\sigma_1, \sigma_2, \sigma_3)$ and $T(\tau_1, \tau_2, \tau_3)$ belong to the same $GL(q, \mathbf{C})$ -orbit.*

Proof. One way is obvious from the transformation law (25) for the triple ratio. For the converse, we assume (as we may) that $\sigma_1 = \tau_1 = ie$ and $\sigma_2 = \tau_2 = -ie$, and set for simplicity $\sigma = \sigma_3$ and $\tau = \tau_3$. Then the assumption implies that $(i\mathbf{1}_q + \sigma_q)(\mathbf{1}_q - i\sigma_q)^{-1}$ and $(i\mathbf{1}_q + \tau_q)(\mathbf{1}_q - i\tau_q)^{-1}$ are in the same $GL(q, \mathbf{C})$ -orbit. By Lemma 2.3, $c(\sigma)$ and $c(\tau)$ are in the same cL -orbit. So σ and τ are in the same L -orbit. \square

Now to give a description of the invariant in terms of Theorem 3.13, we need to define the analog of the function $\arg \det$. For $z_1 \in D$ and $z_2 \in \bar{D}$, the function $k(z_1, z_2) = (\mathbf{1}_q - z_2^* z_1)^{-1}$ is well defined and belongs to $GL(q, \mathbf{C})$. So we can extend the definition of T to the set

$$\tilde{D}_{\top} = \{(z_1, z_2, z_3) \mid z_i \in D \cup S, 1 \leq i \leq 3, z_1 \top' z_2, z_2 \top' z_3, z_3 \top' z_1\},$$

where by definition $z \top' w$ is satisfied if z or w belongs to D , and reduces to the condition $z \top w$ if both z and w belong to S . As \tilde{D}_{\top} is stable by $(z_1, z_2, z_3) \mapsto (tz_1, tz_2, tz_3)$ for $0 \leq t \leq 1$, this is a simply connected set. For $z_1 \in D$, $\det T(z_1, z_1, z_1)$ is a positive real number. So there is a well defined continuous determination of the argument of $\det(T(z_1, z_2, z_3))$ on \tilde{D}_{\top} such that it takes the value 0 whenever $z_1 = z_2 = z_3 \in D$. Denote this determination by $\arg \det T(z_1, z_2, z_3)$. It is clearly invariant under the G -action, and so it defines an invariant for the G -orbits.

On the other hand, let

$$S(z_1, z_2, z_3) = T(z_1, z_2, z_3)^{*^{-1}} T(z_1, z_2, z_3)$$

be the angular matrix associated to $T(z_1, z_2, z_3)$.

THEOREM 4.4. *Let $(\sigma_1, \sigma_2, \sigma_3)$ and (τ_1, τ_2, τ_3) belong to S_{\top}^3 . They belong to the same G -orbit if and only if $S(\sigma_1, \sigma_2, \sigma_3)$ and $S(\tau_1, \tau_2, \tau_3)$ are conjugate under $GL(q, \mathbf{C})$ and $\arg \det T(\sigma_1, \sigma_2, \sigma_3) = \arg \det T(\tau_1, \tau_2, \tau_3)$.*

Proof. This is a direct consequence of Theorem 4.3 and Theorem 3.13.

REMARK 1. Let us consider the case where $q = 1$. The Stiefel manifold is the unit sphere S^{2p-1} in \mathbf{C}^p . The transversality condition $\sigma \top \tau$ just means $\sigma \neq \tau$, as is easily seen from the Cauchy-Schwarz inequality. The triple ratio

is the complex number

$$T(\sigma_1, \sigma_2, \sigma_3) = (1 - \sigma_2^* \sigma_1)^{-1} (1 - \sigma_2^* \sigma_3) (1 - \sigma_1^* \sigma_3)^{-1}.$$

The group $GL(q, \mathbf{C}) \simeq \mathbf{C}^*$ acts on the upper halfplane by $(\lambda, z) \mapsto |\lambda|^2 z$ and so the orbits are described by the argument of the complex number z . So the characteristic invariant in this case is just

$$\arg \left((1 - \sigma_2^* \sigma_1)^{-1} (1 - \sigma_2^* \sigma_3) (1 - \sigma_1^* \sigma_3)^{-1} \right).$$

It is equivalent to the invariant θ considered in [KR]. This invariant, almost in our terms, was known to E. Cartan (see [Ca]).

REMARK 2. Let us consider the case where $p = q$. Then the Stiefel manifold is $U(q)$, and the content of Proposition 4.2 is that for $(\sigma_1, \sigma_2, \sigma_3) \in S_{\top}^3$

$$T(\sigma_1, \sigma_2, \sigma_3) = (1 - \sigma_2^* \sigma_1)^{-1} (1 - \sigma_2^* \sigma_3) (1 - \sigma_1^* \sigma_3)^{-1}$$

is an invertible skew-Hermitian matrix. The orbits of $GL(q, \mathbf{C})$ in its action on nondegenerate Hermitian forms are characterized by the signature. So the characteristic invariant as described in Theorem 4.3 in this case reduces to $\text{sgn } iT(\sigma_1, \sigma_2, \sigma_3)$. As concerns Theorem 4.4, notice that the invariant S is trivial (equal to $-\mathbf{1}_q$), so one is only concerned with the invariant $\arg \det T$. The bounded domain D is of tube type and the description of the invariant through the function $\arg \det$ coincides with the approach of this problem in [CØ], where the invariant was introduced under the name of *generalized Maslov index*.

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