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We can now obtain Corollary 1.4 stated in the Introduction.

**COROLLARY 5.4.** *Let  $G = \langle x_1, \dots, x_k \mid r_1, \dots, r_m \rangle$  be a nonelementary word-hyperbolic group and let  $H \leq G$  be a quasiconvex subgroup of infinite index. Let  $a_n$  be the number of freely reduced words in  $A = \{x_1, \dots, x_k\}^{\pm 1}$  of length  $n$  that represent elements of  $H$ . Let  $b_n$  be the number of all words in  $A$  of length  $n$  that represent elements of  $H$ . Then*

$$\limsup_{n \rightarrow \infty} \sqrt[n]{a_n} < 2k - 1$$

and

$$\limsup_{n \rightarrow \infty} \sqrt[n]{b_n} < 2k.$$

*Proof.* Note that  $k \geq 2$  since  $G$  is nonelementary. Put  $A = \{x_1, \dots, x_k\}$  and  $Y = \Gamma(G, H, A)$ . We choose  $x_0 := H1 \in VY$  as the base-vertex of  $Y$ . Note that  $Y$  is  $2k$ -regular by construction. Also, for any vertex  $x$  of  $Y$  and any word  $w$  in  $A \cup A^{-1}$  there is a unique path in  $Y$  with label  $w$  and origin  $x$ . The definition of Schreier coset graphs also implies that a word  $w$  represents an element of  $H$  if and only if the unique path in  $Y$  with origin  $x_0$  and label  $w$  terminates at  $x_0$ . Therefore  $a_n(Y)$  equals the number of freely reduced words in the alphabet  $A = \{x_1, \dots, x_k\}^{\pm 1}$  of length  $n$  that represent elements of  $H$ . Similarly,  $b_n(Y)$  equals the number of all words in  $A$  of length  $n$  representing elements of  $H$ . By Theorem 1.2,  $Y$  is nonamenable. Hence by Theorem 2.5,  $\alpha(Y) < 2k - 1$  and  $\beta(Y) < 2k$ , as required.

## REFERENCES

- [1] ALONSO, J., T. BRADY, D. COOPER, V. FERLINI, M. LUSTIG, M. MIHALIK, M. SHAPIRO and H. SHORT. Notes on hyperbolic groups. In: *Group Theory from a Geometrical Viewpoint*, Proceedings of the workshop held in Trieste, É. Ghys, A. Haefliger and A. Verjovsky (editors). World Scientific Publishing Co., 1991.
- [2] ANCONA, A. *Théorie du potentiel sur les graphes et les variétés*, École d'été de Probabilités de Saint-Flour XVIII–1988, 1–112. Lecture Notes in Mathematics 1427. Springer, Berlin, 1990.
- [3] ARZHANTSEVA, G. On quasiconvex subgroups of word hyperbolic groups. *Geom. Dedicata* 87 (2001), 191–208.
- [4] BAUMSLAG, G., S. M. GERSTEN, M. SHAPIRO and H. SHORT. Automatic groups and amalgams. *J. Pure Appl. Algebra* 76 (1991), 229–316.

- [5] BARTHOLDI, L. Counting paths in graphs. *L'Enseignement Math.* (2) 45 (1999), 83–131.
- [6] BENJAMINI, I., R. LYONS and O. SCHRAMM. Percolation perturbations in potential theory and random walks. In: *Random Walks and Discrete Potential Theory* (Cortona, 1997), 56–84. Sympos. Math. XXXIX, Cambridge Univ. Press, Cambridge, 1999.
- [7] BENJAMINI, I. and O. SCHRAMM. Every graph with a positive Cheeger constant contains a tree with a positive Cheeger constant. *Geom. Funct. Anal.* 7 (1997), 403–419.
- [8] BESTVINA, M. and M. FEIGHN. A combination theorem for negatively curved groups. *J. Differential Geom.* 35 (1992), 85–101.
- [9] BLOCK, J. and S. WEINBERGER. Aperiodic tilings, positive scalar curvature and amenability of spaces. *J. Amer. Math. Soc.* 5 (1992), 907–918.
- [10] BOROVIK, A., A. G. MYASNIKOV and V. REMESLENNIKOV. Multiplicative measures on free groups. To appear in *Internat. J. Algebra Comput.*
- [11] BOWDITCH, B. Cut points and canonical splittings of hyperbolic groups. *Acta Math.* 180 (1998), 145–186.
- [12] BOWERS, P. Negatively curved graph and planar metrics with applications to type. *Michigan Math. J.* 45 (1998), 31–53.
- [13] BRINKMANN, P. Hyperbolic automorphisms of free groups. *Geom. Funct. Anal.* 10 (2000), 1071–1089.
- [14] CANNON, J. W. The theory of negatively curved spaces and groups. In: *Ergodic Theory, Symbolic Dynamics, and Hyperbolic Spaces* (Trieste, 1989), 315–369. Oxford Sci. Publ., Oxford Univ. Press, New York, 1991.
- [15] CAO, J. Cheeger isoperimetric constants of Gromov-hyperbolic spaces with quasi-poles. *Commun. Contemp. Math.* 2 (2000), 511–533.
- [16] CECCHERINI-SILBERSTEIN, T., R. GRIGORCHUCK and P. DE LA HARPE. Amenability and paradoxical decompositions for pseudogroups and discrete metric spaces (in Russian). *Tr. Mat. Inst. Steklova* 224 (1999). Algebra. Topol. Differ. Uravn. i ikh Prilozh., 68–111; translation in *Proc. Steklov Inst. Math.* 224 (1999), 57–97.
- [17] CHUNG, F. R. K. Laplacians of graphs and Cheeger's inequalities. In: *Combinatorics, Paul Erdős is Eighty*, Vol. 2 (Keszthely, 1993), 157–172. Bolyai Soc. Math. Stud. 2. János Bolyai Math. Soc., Budapest, 1996.
- [18] CHUNG, F. R. K. and K. ODEN. Weighted graph Laplacians and isoperimetric inequalities. *Pacific J. Math.* 192 (2000), 257–273.
- [19] COHEN, J. Cogrowth and amenability of discrete groups. *J. Funct. Anal.* 48 (1982), 301–309.
- [20] COORNAERT, M., T. DELZANT, and A. PAPADOPOULOS. *Géométrie et théorie des groupes. Les groupes hyperboliques de Gromov*. Lecture Notes in Mathematics, 1441. Springer-Verlag, Berlin, 1990.
- [21] DODZIUK, J. Difference equations, isoperimetric inequality and transience of certain random walks. *Trans. Amer. Math. Soc.* 284 (1984), 787–794.
- [22] DUNWOODY, M. and M. SAGEEV. JSJ-splittings for finitely presented groups over slender groups. *Invent. Math.* 135 (1999), 25–44.
- [23] DUNWOODY, M. and E. SWENSON. The algebraic torus theorem. *Invent. Math.* 140 (2000), 605–637.

- [24] ELEK, G. Amenability,  $l_p$ -homologies and translation invariant functionals. *J. Austral. Math. Soc. Ser. A* 65 (1998), 111–119.
- [25] EPSTEIN, D., J. CANNON, D. HOLT, S. LEVY, M. PATERSON and W. THURSTON. *Word Processing in Groups*. Jones and Bartlett, Boston, 1992.
- [26] EPSTEIN, D. and D. HOLT. Efficient computation in word-hyperbolic groups. In : *Computational and Geometric Aspects of Modern Algebra (Edinburgh, 1998)*, 66–77. LMS Lecture Note Ser. 275. Cambridge Univ. Press, Cambridge, 2000.
- [27] FOORD, R. Automaticity and growth in certain classes of groups and monoids. PhD Thesis, Warwick University, 2000.
- [28] FUJIWARA, K. and P. PAPASOGLU. JSJ decompositions and complexes of groups. Preprint, 1996.
- [29] GERASIMOV, V.N. Semi-splittings of groups and actions on cubings. In : *Algebra, Geometry, Analysis and Mathematical Physics (Novosibirsk, 1996)*, 91–109, 190. Izdat. Ross. Akad. Nauk Sib. Otd. Inst. Mat., Novosibirsk, 1997.
- [30] GERL, P. Amenable groups and amenable graphs. In : *Harmonic Analysis (Luxembourg, 1987)*, 181–190. Lecture Notes in Mathematics 1359. Springer, Berlin, 1988.
- [31] GERSTEN, S.M. and H. SHORT. Rational subgroups of biautomatic groups. *Ann. of Math.* (2) 134 (1991), 125–158.
- [32] GHYS, E. and P. DE LA HARPE (eds). *Sur les groupes hyperboliques d'après Mikhael Gromov*. Progress in Mathematics series 83. Birkhäuser, 1990.
- [33] GITIK, R. On the combination theorem for negatively curved groups. *Internat. J. Algebra Comput.* 6 (1996), 751–760.
- [34] —— On quasiconvex subgroups of negatively curved groups. *J. Pure Appl. Algebra* 119 (1997), 155–169.
- [35] —— On the profinite topology on negatively curved groups. *J. Algebra* 219 (1999), 80–86.
- [36] —— Doubles of groups and hyperbolic LERF 3-manifolds. *Ann. of Math.* (2) 150 (1999), 775–806.
- [37] —— Tameness and geodesic cores of subgroups. *J. Austral. Math. Soc. Ser. A* 69 (2000), 153–16.
- [38] GITIK, R., M. MITRA, E. RIPS and M. SAGEEV. Widths of subgroups. *Trans. Amer. Math. Soc.* 350 (1998), 321–329.
- [39] GRIGORCHUK, R.I. Symmetrical random walks on discrete groups. In : *Multi-component Random Systems*, 285–325. Adv. Probab. Related Topics 6. Dekker, New York, 1998.
- [40] GROMOV, M. Hyperbolic groups. In : *Essays in Group Theory*, (S. M. Gersten, ed.), 75–263. Math. Sci. Res. Inst. Publ. 8, 1987.
- [41] —— Asymptotic invariants of infinite groups. In : *Geometric Group Theory*, Vol. 2 (Sussex, 1991), 1–295. LMS Lecture Note Ser. 182. Cambridge Univ. Press, Cambridge, 1993.
- [42] HOLT, D. Automatic groups, subgroups and cosets. In : *The Epstein Birthday Schrift*, 249–260. Geom. Topol. Monogr. 1. Geom. Topol., Coventry, 1998.

- [43] KAIMANOVICH, V. Equivalence relations with amenable leaves need not be amenable. In: *Topology, Ergodic Theory, Real Algebraic Geometry*, 151–166. Amer. Math. Soc. Transl. Ser. 2 202. Amer. Math. Soc., Providence (R.I.), 2001.
- [44] KAIMANOVICH, V. and W. WOESS. The Dirichlet problem at infinity for random walks on graphs with a strong isoperimetric inequality. *Probab. Theory Related Fields* 91 (1992), 445–466.
- [45] KAPOVICH, I. Detecting quasiconvexity: algorithmic aspects. In: *Geometric and Computational Perspectives on Infinite Groups (Minneapolis, MN and New Brunswick, NJ, 1994)*, 91–99. DIMACS Ser. Discrete Math. Theoret. Comput. Sci. 25. Amer. Math. Soc., Providence (R.I.), 1996.
- [46] —— Quasiconvexity and amalgams. *Internat. J. Algebra Comput.* 7 (1997), 771–811.
- [47] —— A non-quasiconvexity embedding theorem for word-hyperbolic groups. *Math. Proc. Cambridge Philos. Soc.* 127 (1999), 461–486.
- [48] —— The combination theorem and quasiconvexity. *Internat. J. Algebra Comput.* 11 (2001), 185–216.
- [49] —— The geometry of relative Cayley graphs for subgroups of hyperbolic groups. Preprint, 2002.
- [50] KAPOVICH, I., A. MYASNIKOV, P. SCHUPP and V. SHPILRAIN. Generic-case complexity, decision problems in group theory and random walks. Preprint, 2002.
- [51] KAPOVICH, I. and H. SHORT. Greenberg’s theorem for quasiconvex subgroups of word hyperbolic groups. *Canad. J. Math.* 48 (1996), 1224–1244.
- [52] LUBOTZKY, A. *Discrete Groups, Expanding Graphs and Invariant Measures*. (With an appendix by Jonathan D. Rogawski.) Progress in Mathematics 125. Birkhäuser Verlag, Basel, 1994.
- [53] MIHALIK, M. Group extensions and tame pairs. *Trans. Amer. Math. Soc.* 351 (1999), 1095–1107.
- [54] MIHALIK, M. and W. TOWLE. Quasiconvex subgroups of negatively curved groups. *J. Pure Appl. Algebra* 95 (1994), 297–301.
- [55] MITRA, M. Cannon-Thurston maps for trees of hyperbolic metric spaces. *J. Differential Geom.* 48 (1998), 135–164.
- [56] NEUMANN, B. H. Groups covered by finitely many cosets. *Publ. Math. Debrecen* 3 (1954), 227–242.
- [57] REEVES, L. Rational subgroups of cubed 3-manifold groups. *Michigan Math. J.* 42 (1995), 109–126.
- [58] RIPS, E. Subgroups of small cancellation groups. *Bull. London Math. Soc.* 14 (1982), 45–47.
- [59] RIPS, E. and Z. SELA. Cyclic splittings of finitely presented groups and the canonical JSJ decomposition. *Ann. of Math.* (2) 146 (1997), 53–109.
- [60] SAGEEV, M. Ends of group pairs and non-positively curved cube complexes. *Proc. London Math. Soc.* (3) 71 (1995), 585–617.
- [61] —— Codimension-1 subgroups and splittings of groups. *J. Algebra* 189 (1997), 377–389.
- [62] SCHONMANN, R. Multiplicity of phase transitions and mean-field criticality on highly non-amenable graphs. *Comm. Math. Phys.* 219 (2001), 271–322.

- [63] SCOTT, G. P. and G. A. SWARUP. An algebraic annulus theorem. *Pacific J. Math.* 196 (2000), 461–506.
- [64] SCOTT, G. P. and G. A. SWARUP. Canonical splittings of groups and 3-manifolds. *Trans. Amer. Math. Soc.* 353 (2001), 4973–5001.
- [65] SELA, Z. Structure and rigidity in (Gromov) hyperbolic groups and discrete groups in rank 1 Lie groups. II. *Geom. Funct. Anal.* 7 (1997), 561–593.
- [66] SHALOM, Y. Random ergodic theorems, invariant means and unitary representation. In: *Lie Groups and Ergodic Theory (Mumbai, 1996)*, 273–314. Tata Inst. Fund. Res. Stud. Math. 14. Tata Inst. Fund. Res., Bombay, 1998.
- [67] —— Expander graphs and amenable quotients. In: *Emerging Applications of Number Theory (Minneapolis, 1996)*, 571–581. IMA Vol. Math. Appl. 109. Springer, New York, 1999.
- [68] SHORT, H. Quasiconvexity and a theorem of Howson's. In: *Group Theory from a Geometrical Viewpoint (Trieste, 1990)*, 168–176. World Sci. Publishing, River Edge (N.J.), 1991.
- [69] SWARUP, G. A. Geometric finiteness and rationality. *J. Pure Appl. Algebra* 86 (1993), 327–333.
- [70] —— Proof of a weak hyperbolization theorem. *Quarterly J. Math.* 51 (2000), 529–533.
- [71] WOESS, W. Random walks on infinite graphs and groups – a survey on selected topics. *Bull. London Math. Soc.* 26 (1994), 1–60.
- [72] —— *Random Walks on Infinite Graphs and Groups*. Cambridge Tracts in Mathematics 138. Cambridge University Press, Cambridge, 2000.
- [73] ZUK, A. On property (T) for discrete groups. Preprint, 2001.

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