

Zeitschrift:	L'Enseignement Mathématique
Herausgeber:	Commission Internationale de l'Enseignement Mathématique
Band:	48 (2002)
Heft:	3-4: L'ENSEIGNEMENT MATHÉMATIQUE
 Artikel:	THE NONAMENABILITY OF SCHREIER GRAPHS FOR INFINITE INDEX QUASICONVEX SUBGROUPS OF HYPERBOLIC GROUPS
Autor:	Kapovich, Ilya
Kapitel:	5. PROOF OF THE MAIN RESULT
DOI:	https://doi.org/10.5169/seals-66081

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 25.02.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

4. Suppose $H \leq G$ is generated by a finite set Q inducing the word-metric d_Q on H . Then H is quasiconvex in G if and only if there is a $C > 0$ such that for any $h_1, h_2 \in H$

$$d_Q(h_1, h_2) \leq Cd_A(h_1, h_2)$$

(see [20, 32, 4, 31]).

5. The set \mathcal{L} of all A -geodesic words is a regular language that provides a bi-automatic structure for G . Moreover, a subgroup $H \leq G$ is quasiconvex if and only if H is \mathcal{L} -rational, that is the set $\mathcal{L}_H = \{w \in \mathcal{L} \mid \overline{w} \in H\}$ is a regular language [31].

6. If $H_1, H_2 \leq G$ are quasiconvex, then $H_1 \cap H_2 \leq G$ is quasiconvex [68].

7. [51, 46] Let $C \leq B \leq G$ where B is quasiconvex in G (and hence B is hyperbolic) and C is quasiconvex in B . Then C is quasiconvex in G [51, 46].

8. Let $C \leq B \leq G$ where C is quasiconvex in G and where B is word-hyperbolic. Then C is quasiconvex in B [51, 46].

9. Suppose $H \leq G$ is an infinite quasiconvex subgroup. Then H has finite index in its commensurator $\text{Comm}_G(H)$ (see [51]), where $\text{Comm}_G(H) := \{g \in G \mid [H : g^{-1}Hg \cap H] < \infty \text{ and } [g^{-1}Hg : g^{-1}Hg \cap H] < \infty\}$.

Part 1 of the above proposition implies that a nonelementary subgroup of a hyperbolic group is nonamenable.

5. PROOF OF THE MAIN RESULT

Let G be a nonelementary word-hyperbolic group with a finite generating set A . Let $X = \Gamma(G, A)$ be the Cayley graph of G with the word metric d_A . Let $\delta \geq 1$ be an integer such that the space $(\Gamma(G, A), d_A)$ is δ -hyperbolic. Let $H \leq G$ be a quasiconvex subgroup of infinite index in G . These conventions, unless specified otherwise, will be fixed for the remainder of the paper.

We shall need the following useful fact:

LEMMA 5.1. *There exists an integer constant $K = K(G, H, A) > 0$ with the following properties.*

Assume $g \in G$ is shortest with respect to d_A in the coset class Hg . Then for any $h \in H$ we have $(g, h)_1 \leq K$ (and hence $(g, H)_1 \leq K$).

Proof. The conclusion of Lemma 5.1 follows directly from the proofs of Lemma 4.1 and Lemma 4.5 of [4]. We will present the argument for completeness. For the hyperbolic space $X = \Gamma(G, A)$ choose $\delta' \geq 0$ as in part 2 of Proposition 3.3. Let $\epsilon \geq 0$ be such that H is an ϵ -quasiconvex subset of X .

Let $g \in G$ be a shortest element of Hg , so that for any $h \in H$ we have $|hg|_A \leq |g|_A$. We claim that $(h, g)_1 \leq \epsilon + \delta'$ for any $h \in H$.

Suppose not, that is $(h, g)_1 > \epsilon + \delta'$ for some $h \in H$. Consider two geodesic segments $[1, g]$ and $[1, h]$ in X and let $t \in [1, h]$, $s \in [1, g]$ be such that $d_A(1, s) = d_A(1, t) = (h, g)_1$. Thus $d_A(s, t) \leq \delta'$ by the choice of δ' . Since H is ϵ -quasiconvex in X , there is $h' \in H$ such that $d_A(t, h') \leq \epsilon$. Then

$$\begin{aligned} |(h')^{-1}g|_A &= d_A(h', g) \leq d_A(h', t) + d_A(t, s) + d_A(s, g) \\ &\leq \epsilon + \delta + |g|_A - (h, g)_1 < |g|_A, \end{aligned}$$

which contradicts the assumption that g is shortest in Hg .

LEMMA 5.2. *Let $T_1, T_2 > 0$ be some positive numbers. Let $g \in G$ be such that $(g, H)_1 \leq T_1$ and $|g|_A > T_1 + T_2 + \delta$. Let $f \in G$ be such that $|f|_A \leq T_2$. Then $(gf, H)_1 \leq T_1 + \delta$.*

Proof. Note that $|g|_A = (g, gf)_1 + (1, gf)_g$. Since $(1, gf)_g \leq d(g, gf) = |f|_A \leq T_2$, we conclude that

$$(g, gf)_1 = |g|_A - (1, gf)_g > T_1 + T_2 + \delta - T_2 = T_1 + \delta.$$

Therefore for any $h \in H$ we have

$$T_1 + \delta \geq (g, h)_1 + \delta \geq \min\{(g, gf)_1, (gf, h)_1\}$$

and hence $(gf, h)_1 \leq T_1 + \delta$ because $(g, gf)_1 > T_1 + \delta$. Since $h \in H$ was arbitrary, this means that $(gf, H)_1 \leq T_1 + \delta$.

LEMMA 5.3. *Suppose $g_1, g_2 \in G$ are such that $Hg_1 = Hg_2$. Then there is $h \in H$ such that $hg_1 = g_2$ and that*

$$|h|_A \leq (g_1, H)_1 + (g_2, H)_1.$$

Proof. Since $Hg_1 = Hg_2$, there is $h \in H$ with $hg_1 = g_2$. Hence

$$|h|_A = (h, g_2)_1 + (1, hg_1)_h = (h, g_2)_1 + (h^{-1}, g_1)_1 \leq (g_2, H)_1 + (g_1, H)_1.$$

Proof of Theorem 1.2. Let $K = K(G, H, A) > 0$ be the constant provided by Lemma 5.1. Put $Y = \Gamma(G, H, A)$. Thus Y is a connected $2m$ -regular infinite graph, where m is the number of elements in A . Denote the simplicial metric on Y by d_Y .

Let N be the number of all elements $g \in G$ with $|g|_A \leq 2K + 2\delta$. In particular Y has at most N vertices within distance $2K + 2\delta$ of the coset $H1 \in VY$.

Since G is nonelementary word-hyperbolic and thus nonamenable, the Cayley graph $X = \Gamma(G, A)$ is nonamenable. By part 4 of Proposition 2.3 there is a constant $k' > 0$ such that for any finite nonempty subset S of G the k' -neighborhood of S in X has at least $4N|S|$ vertices. Let N_1 be the number of elements of G of length at most $K + \delta + k'$. Choose $k'' > 1$ such that for any vertex $Hg \in VY$ with $d_Y(H1, Hg) \leq K + \delta + k'$ the k'' -neighborhood of Hg has at least $4N_1$ vertices. Such k'' exists since by assumption $[G : H] = \infty$ and hence the graph Y is infinite. Set $k := \max\{k', k''\}$.

Suppose now that $F \subset VY$ is a finite nonempty subset. Write $F = F_1 \sqcup F_2$ where F_1 is the intersection of F with the closed ball of radius $K + \delta + k'$ in Y .

If $|F_1| \geq |F|/2$, then $|F| \leq 2N_1$ and the k -neighborhood of F in Y has at least $4N_1 \geq 2|F|$ vertices. Suppose now that $|F_1| < |F|/2$, so that $|F_2| \geq |F|/2$. Then

$$F_2 = \{Hg_1, \dots, Hg_t\}$$

where $|F_2| = t$ and where each $g_i \in G$ is shortest in Hg_i with $|g_i|_A > K + \delta + k'$. By Lemma 5.1 $(g_i, H)_1 \leq K$. By Lemma 5.2 for any $f \in G$ with $|f|_A \leq k'$ and for each $i = 1, \dots, t$ we have $(g_i f, H)_1 \leq K + \delta$.

Let $S := \{g_1, \dots, g_t\}$ and let S' be the set of all vertices of X contained in the k' -neighborhood of S in X . By the choice of k' we have $|S'| \geq 4N|S| = 4N|F_2|$. On the other hand, Lemma 5.3 implies that if $g, g' \in S'$ are such that $Hg = Hg'$ then $hg = g'$ for some $h \in H$ with $|h|_A \leq 2K + 2\delta$. By the choice of N this means that the set $F' := \{Hg \mid g \in S'\}$ contains at least

$$|S'|/N = 4N|F_2|/N = 4|F_2| \geq 2|F|$$

distinct elements. However, F' is obviously contained in the k -neighborhood of F in Y .

We have verified that for any finite nonempty subset $F \subseteq VY$ the k -neighborhood of F in Y contains at least $2|F|$ vertices. By the Doubling Condition (part 3 of Proposition 2.3) this implies that Y is nonamenable.

We can now obtain Corollary 1.4 stated in the Introduction.

COROLLARY 5.4. *Let $G = \langle x_1, \dots, x_k \mid r_1, \dots, r_m \rangle$ be a nonelementary word-hyperbolic group and let $H \leq G$ be a quasiconvex subgroup of infinite index. Let a_n be the number of freely reduced words in $A = \{x_1, \dots, x_k\}^{\pm 1}$ of length n that represent elements of H . Let b_n be the number of all words in A of length n that represent elements of H . Then*

$$\limsup_{n \rightarrow \infty} \sqrt[n]{a_n} < 2k - 1$$

and

$$\limsup_{n \rightarrow \infty} \sqrt[n]{b_n} < 2k.$$

Proof. Note that $k \geq 2$ since G is nonelementary. Put $A = \{x_1, \dots, x_k\}$ and $Y = \Gamma(G, H, A)$. We choose $x_0 := H1 \in VY$ as the base-vertex of Y . Note that Y is $2k$ -regular by construction. Also, for any vertex x of Y and any word w in $A \cup A^{-1}$ there is a unique path in Y with label w and origin x . The definition of Schreier coset graphs also implies that a word w represents an element of H if and only if the unique path in Y with origin x_0 and label w terminates at x_0 . Therefore $a_n(Y)$ equals the number of freely reduced words in the alphabet $A = \{x_1, \dots, x_k\}^{\pm 1}$ of length n that represent elements of H . Similarly, $b_n(Y)$ equals the number of all words in A of length n representing elements of H . By Theorem 1.2, Y is nonamenable. Hence by Theorem 2.5, $\alpha(Y) < 2k - 1$ and $\beta(Y) < 2k$, as required.

REFERENCES

- [1] ALONSO, J., T. BRADY, D. COOPER, V. FERLINI, M. LUSTIG, M. MIHALIK, M. SHAPIRO and H. SHORT. Notes on hyperbolic groups. In: *Group Theory from a Geometrical Viewpoint*, Proceedings of the workshop held in Trieste, É. Ghys, A. Haefliger and A. Verjovsky (editors). World Scientific Publishing Co., 1991.
- [2] ANCONA, A. *Théorie du potentiel sur les graphes et les variétés*, École d'été de Probabilités de Saint-Flour XVIII–1988, 1–112. Lecture Notes in Mathematics 1427. Springer, Berlin, 1990.
- [3] ARZHANTSEVA, G. On quasiconvex subgroups of word hyperbolic groups. *Geom. Dedicata* 87 (2001), 191–208.
- [4] BAUMSLAG, G., S. M. GERSTEN, M. SHAPIRO and H. SHORT. Automatic groups and amalgams. *J. Pure Appl. Algebra* 76 (1991), 229–316.