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INDEX QUASICONVEX SUBGROUPS OF HYPERBOLIC GROUPS

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4. Suppose  $H \leq G$  is generated by a finite set Q inducing the word-metric  $d_Q$  on H. Then H is quasiconvex in G if and only if there is a C > 0 such that for any  $h_1, h_2 \in H$ 

$$d_Q(h_1,h_2) \leq Cd_A(h_1,h_2)$$

(see [20, 32, 4, 31]).

- 5. The set  $\mathcal{L}$  of all A-geodesic words is a regular language that provides a bi-automatic structure for G. Moreover, a subgroup  $H \leq G$  is quasiconvex if and only if H is  $\mathcal{L}$ -rational, that is the set  $\mathcal{L}_H = \{w \in \mathcal{L} \mid \overline{w} \in H\}$  is a regular language [31].
- 6. If  $H_1, H_2 \leq G$  are quasiconvex, then  $H_1 \cap H_2 \leq G$  is quasiconvex [68].
- 7. [51, 46] Let  $C \leq B \leq G$  where B is quasiconvex in G (and hence B is hyperbolic) and C is quasiconvex in B. Then C is quasiconvex in G [51, 46].
- 8. Let  $C \leq B \leq G$  where C is quasiconvex in G and where B is word-hyperbolic. Then C is quasiconvex in B [51, 46].
- 9. Suppose  $H \leq G$  is an infinite quasiconvex subgroup. Then H has finite index in its commensurator  $Comm_G(H)$  (see [51]), where  $Comm_G(H) := \{g \in G \mid [H : g^{-1}Hg \cap H] < \infty \text{ and } [g^{-1}Hg : g^{-1}Hg \cap H] < \infty \}.$

Part 1 of the above proposition implies that a nonelementary subgroup of a hyperbolic group is nonamenable.

# 5. PROOF OF THE MAIN RESULT

Let G be a nonelementary word-hyperbolic group with a finite generating set A. Let  $X = \Gamma(G,A)$  be the Cayley graph of G with the word metric  $d_A$ . Let  $\delta \geq 1$  be an integer such that the space  $(\Gamma(G,A),d_A)$  is  $\delta$ -hyperbolic. Let  $H \leq G$  be a quasiconvex subgroup of infinite index in G. These conventions, unless specified otherwise, will be fixed for the remainder of the paper.

We shall need the following useful fact:

LEMMA 5.1. There exists an integer constant K = K(G, H, A) > 0 with the following properties.

Assume  $g \in G$  is shortest with respect to  $d_A$  in the coset class Hg. Then for any  $h \in H$  we have  $(g,h)_1 \leq K$  (and hence  $(g,H)_1 \leq K$ ).

*Proof.* The conclusion of Lemma 5.1 follows directly from the proofs of Lemma 4.1 and Lemma 4.5 of [4]. We will present the argument for completeness. For the hyperbolic space  $X = \Gamma(G,A)$  choose  $\delta' \geq 0$  as in part 2 of Proposition 3.3. Let  $\epsilon \geq 0$  be such that H is an  $\epsilon$ -quasiconvex subset of X.

Let  $g \in G$  be a shortest element of Hg, so that for any  $h \in H$  we have  $|hg|_A \leq |g|_A$ . We claim that  $(h,g)_1 \leq \epsilon + \delta'$  for any  $h \in H$ .

Suppose not, that is  $(h,g)_1 > \epsilon + \delta'$  for some  $h \in H$ . Consider two geodesic segments [1,g] and [1,h] in X and let  $t \in [1,h]$ ,  $s \in [1,g]$  be such that  $d_A(1,s) = d_A(1,t) = (h,g)_1$ . Thus  $d_A(s,t) \leq \delta'$  by the choice of  $\delta'$ . Since H is  $\epsilon$ -quasiconvex in X, there is  $h' \in H$  such that  $d_A(t,h') \leq \epsilon$ . Then

$$|(h')^{-1}g|_{A} = d_{A}(h',g) \le d_{A}(h',t) + d_{A}(t,s) + d_{A}(s,g)$$
  
$$\le \epsilon + \delta + |g|_{A} - (h,g)_{1} < |g|_{A},$$

which contradicts the assumption that g is shortest in Hg.

LEMMA 5.2. Let  $T_1, T_2 > 0$  be some positive numbers. Let  $g \in G$  be such that  $(g, H)_1 \leq T_1$  and  $|g|_A > T_1 + T_2 + \delta$ . Let  $f \in G$  be such that  $|f|_A \leq T_2$ . Then  $(gf, H)_1 \leq T_1 + \delta$ .

*Proof.* Note that  $|g|_A = (g, gf)_1 + (1, gf)_g$ . Since  $(1, gf)_g \le d(g, gf) = |f|_A \le T_2$ , we conclude that

$$(g, gf)_1 = |g|_A - (1, gf)_g > T_1 + T_2 + \delta - T_2 = T_1 + \delta.$$

Therefore for any  $h \in H$  we have

$$T_1 + \delta \ge (g, h)_1 + \delta \ge \min\{(g, gf)_1, (gf, h)_1\}$$

and hence  $(gf,h)_1 \leq T_1 + \delta$  because  $(g,gf)_1 > T_1 + \delta$ . Since  $h \in H$  was arbitrary, this means that  $(gf,H)_1 \leq T_1 + \delta$ .

LEMMA 5.3. Suppose  $g_1, g_2 \in G$  are such that  $Hg_1 = Hg_2$ . Then there is  $h \in H$  such that  $hg_1 = g_2$  and that

$$|h|_A \leq (g_1, H)_1 + (g_2, H)_1$$
.

*Proof.* Since  $Hg_1 = Hg_2$ , there is  $h \in H$  with  $hg_1 = g_2$ . Hence

$$|h|_A = (h, g_2)_1 + (1, hg_1)_h = (h, g_2)_1 + (h^{-1}, g_1)_1 \le (g_2, H)_1 + (g_1, H)_1.$$

Proof of Theorem 1.2. Let K = K(G, H, A) > 0 be the constant provided by Lemma 5.1. Put  $Y = \Gamma(G, H, A)$ . Thus Y is a connected 2m-regular infinite graph, where m is the number of elements in A. Denote the simplicial metric on Y by  $d_Y$ .

Let N be the number of all elements  $g \in G$  with  $|g|_A \le 2K + 2\delta$ . In particular Y has at most N vertices within distance  $2K + 2\delta$  of the coset  $H1 \in VY$ .

Since G is nonelementary word-hyperbolic and thus nonamenable, the Cayley graph  $X = \Gamma(G,A)$  is nonamenable. By part 4 of Proposition 2.3 there is a constant k' > 0 such that for any finite nonempty subset S of G the k'-neighborhood of S in X has at least 4N|S| vertices. Let  $N_1$  be the number of elements of G of length at most  $K+\delta+k'$ . Choose k''>1 such that for any vertex  $Hg \in VY$  with  $d_Y(H1,Hg) \leq K+\delta+k'$  the k''-neighborhood of Hg has at least  $4N_1$  vertices. Such k'' exists since by assumption  $[G:H] = \infty$  and hence the graph Y is infinite. Set  $k := \max\{k',k''\}$ .

Suppose now that  $F \subset VY$  is a finite nonempy subset. Write  $F = F_1 \sqcup F_2$  where  $F_1$  is the intersection of F with the closed ball of radius  $K + \delta + k'$  in Y.

If  $|F_1| \ge |F|/2$ , then  $|F| \le 2N_1$  and the *k*-neighborhood of *F* in *Y* has at least  $4N_1 \ge 2|F|$  vertices. Suppose now that  $|F_1| < |F|/2$ , so that  $|F_2| \ge |F|/2$ . Then

$$F_2 = \{Hg_1, \dots, Hg_t\}$$

where  $|F_2| = t$  and where each  $g_i \in G$  is shortest in  $Hg_i$  with  $|g_i|_A > K + \delta + k'$ . By Lemma 5.1  $(g_i, H)_1 \leq K$ . By Lemma 5.2 for any  $f \in G$  with  $|f|_A \leq k'$  and for each  $i = 1, \ldots, t$  we have  $(g_i f, H)_1 \leq K + \delta$ .

Let  $S:=\{g_1,\ldots,g_t\}$  and let S' be the set of all vertices of X contained in the k'-neighborhood of S in X. By the choice of k' we have  $|S'| \geq 4N|S| = 4N|F_2|$ . On the other hand, Lemma 5.3 implies that if  $g,g' \in S'$  are such that Hg = Hg' then hg = g' for some  $h \in H$  with  $|h|_A \leq 2K + 2\delta$ . By the choice of N this means that the set  $F' := \{Hg \mid g \in S'\}$  contains at least

$$|S'|/N = 4N|F_2|/N = 4|F_2| \ge 2|F|$$

distinct elements. However, F' is obviously contained in the k-neighborhood of F in Y.

We have verified that for any finite nonempy subset  $F \subseteq VY$  the k-neighborhood of F in Y contains at least 2|F| vertices. By the Doubling Condition (part 3 of Proposition 2.3) this implies that Y is nonamenable.

We can now obtain Corollary 1.4 stated in the Introduction.

COROLLARY 5.4. Let  $G = \langle x_1, \ldots, x_k \mid r_1, \ldots, r_m \rangle$  be a nonelementary word-hyperbolic group and let  $H \leq G$  be a quasiconvex subgroup of infinite index. Let  $a_n$  be the number of freely reduced words in  $A = \{x_1, \ldots, x_k\}^{\pm 1}$  of length n that represent elements of H. Let  $b_n$  be the number of all words in A of length n that represent elements of H. Then

$$\limsup_{n\to\infty}\sqrt[n]{a_n}<2k-1$$

and

$$\limsup_{n\to\infty} \sqrt[n]{b_n} < 2k.$$

Proof. Note that  $k \ge 2$  since G is nonelementary. Put  $A = \{x_1, \ldots, x_k\}$  and  $Y = \Gamma(G, H, A)$ . We choose  $x_0 := H1 \in VY$  as the base-vertex of Y. Note that Y is 2k-regular by construction. Also, for any vertex x of Y and any word w in  $A \cup A^{-1}$  there is a unique path in Y with label w and origin x. The definition of Schreier coset graphs also implies that a word w represents an element of H if and only if the unique path in Y with origin  $x_0$  and label w terminates at  $x_0$ . Therefore  $a_n(Y)$  equals the number of freely reduced words in the alphabet  $A = \{x_1, \ldots, x_k\}^{\pm 1}$  of length n that represent elements of H. Similarly,  $b_n(Y)$  equals the number of all words in A of length n representing elements of H. By Theorem 1.2, Y is nonamenable. Hence by Theorem 2.5,  $\alpha(Y) < 2k - 1$  and  $\beta(Y) < 2k$ , as required.

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