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4. Suppose $H \leq G$ is generated by a finite set Q inducing the word-metric d_Q on H . Then H is quasiconvex in G if and only if there is a $C > 0$ such that for any $h_1, h_2 \in H$

$$d_Q(h_1, h_2) \leq Cd_A(h_1, h_2)$$

(see [20, 32, 4, 31]).

5. The set \mathcal{L} of all A -geodesic words is a regular language that provides a bi-automatic structure for G . Moreover, a subgroup $H \leq G$ is quasiconvex if and only if H is \mathcal{L} -rational, that is the set $\mathcal{L}_H = \{w \in \mathcal{L} \mid \overline{w} \in H\}$ is a regular language [31].
6. If $H_1, H_2 \leq G$ are quasiconvex, then $H_1 \cap H_2 \leq G$ is quasiconvex [68].
7. [51, 46] Let $C \leq B \leq G$ where B is quasiconvex in G (and hence B is hyperbolic) and C is quasiconvex in B . Then C is quasiconvex in G [51, 46].
8. Let $C \leq B \leq G$ where C is quasiconvex in G and where B is word-hyperbolic. Then C is quasiconvex in B [51, 46].
9. Suppose $H \leq G$ is an infinite quasiconvex subgroup. Then H has finite index in its commensurator $\text{Comm}_G(H)$ (see [51]), where $\text{Comm}_G(H) := \{g \in G \mid [H : g^{-1}Hg \cap H] < \infty \text{ and } [g^{-1}Hg : g^{-1}Hg \cap H] < \infty\}$.

Part 1 of the above proposition implies that a nonelementary subgroup of a hyperbolic group is nonamenable.

5. PROOF OF THE MAIN RESULT

Let G be a nonelementary word-hyperbolic group with a finite generating set A . Let $X = \Gamma(G, A)$ be the Cayley graph of G with the word metric d_A . Let $\delta \geq 1$ be an integer such that the space $(\Gamma(G, A), d_A)$ is δ -hyperbolic. Let $H \leq G$ be a quasiconvex subgroup of infinite index in G . These conventions, unless specified otherwise, will be fixed for the remainder of the paper.

We shall need the following useful fact:

LEMMA 5.1. *There exists an integer constant $K = K(G, H, A) > 0$ with the following properties.*

Assume $g \in G$ is shortest with respect to d_A in the coset class Hg . Then for any $h \in H$ we have $(g, h)_1 \leq K$ (and hence $(g, H)_1 \leq K$).

Proof. The conclusion of Lemma 5.1 follows directly from the proofs of Lemma 4.1 and Lemma 4.5 of [4]. We will present the argument for completeness. For the hyperbolic space $X = \Gamma(G, A)$ choose $\delta' \geq 0$ as in part 2 of Proposition 3.3. Let $\epsilon \geq 0$ be such that H is an ϵ -quasiconvex subset of X .

Let $g \in G$ be a shortest element of Hg , so that for any $h \in H$ we have $|hg|_A \leq |g|_A$. We claim that $(h, g)_1 \leq \epsilon + \delta'$ for any $h \in H$.

Suppose not, that is $(h, g)_1 > \epsilon + \delta'$ for some $h \in H$. Consider two geodesic segments $[1, g]$ and $[1, h]$ in X and let $t \in [1, h]$, $s \in [1, g]$ be such that $d_A(1, s) = d_A(1, t) = (h, g)_1$. Thus $d_A(s, t) \leq \delta'$ by the choice of δ' . Since H is ϵ -quasiconvex in X , there is $h' \in H$ such that $d_A(t, h') \leq \epsilon$. Then

$$\begin{aligned} |(h')^{-1}g|_A &= d_A(h', g) \leq d_A(h', t) + d_A(t, s) + d_A(s, g) \\ &\leq \epsilon + \delta' + |g|_A - (h, g)_1 < |g|_A, \end{aligned}$$

which contradicts the assumption that g is shortest in Hg .

LEMMA 5.2. *Let $T_1, T_2 > 0$ be some positive numbers. Let $g \in G$ be such that $(g, H)_1 \leq T_1$ and $|g|_A > T_1 + T_2 + \delta$. Let $f \in G$ be such that $|f|_A \leq T_2$. Then $(gf, H)_1 \leq T_1 + \delta$.*

Proof. Note that $|g|_A = (g, gf)_1 + (1, gf)_g$. Since $(1, gf)_g \leq d(g, gf) = |f|_A \leq T_2$, we conclude that

$$(g, gf)_1 = |g|_A - (1, gf)_g > T_1 + T_2 + \delta - T_2 = T_1 + \delta.$$

Therefore for any $h \in H$ we have

$$T_1 + \delta \geq (g, h)_1 + \delta \geq \min\{(g, gf)_1, (gf, h)_1\}$$

and hence $(gf, h)_1 \leq T_1 + \delta$ because $(g, gf)_1 > T_1 + \delta$. Since $h \in H$ was arbitrary, this means that $(gf, H)_1 \leq T_1 + \delta$.

LEMMA 5.3. *Suppose $g_1, g_2 \in G$ are such that $Hg_1 = Hg_2$. Then there is $h \in H$ such that $hg_1 = g_2$ and that*

$$|h|_A \leq (g_1, H)_1 + (g_2, H)_1.$$

Proof. Since $Hg_1 = Hg_2$, there is $h \in H$ with $hg_1 = g_2$. Hence

$$|h|_A = (h, g_2)_1 + (1, hg_1)_h = (h, g_2)_1 + (h^{-1}, g_1)_1 \leq (g_2, H)_1 + (g_1, H)_1.$$

Proof of Theorem 1.2. Let $K = K(G, H, A) > 0$ be the constant provided by Lemma 5.1. Put $Y = \Gamma(G, H, A)$. Thus Y is a connected $2m$ -regular infinite graph, where m is the number of elements in A . Denote the simplicial metric on Y by d_Y .

Let N be the number of all elements $g \in G$ with $|g|_A \leq 2K + 2\delta$. In particular Y has at most N vertices within distance $2K + 2\delta$ of the coset $H1 \in VY$.

Since G is nonelementary word-hyperbolic and thus nonamenable, the Cayley graph $X = \Gamma(G, A)$ is nonamenable. By part 4 of Proposition 2.3 there is a constant $k' > 0$ such that for any finite nonempty subset S of G the k' -neighborhood of S in X has at least $4N|S|$ vertices. Let N_1 be the number of elements of G of length at most $K + \delta + k'$. Choose $k'' > 1$ such that for any vertex $Hg \in VY$ with $d_Y(H1, Hg) \leq K + \delta + k'$ the k'' -neighborhood of Hg has at least $4N_1$ vertices. Such k'' exists since by assumption $[G : H] = \infty$ and hence the graph Y is infinite. Set $k := \max\{k', k''\}$.

Suppose now that $F \subset VY$ is a finite nonempty subset. Write $F = F_1 \sqcup F_2$ where F_1 is the intersection of F with the closed ball of radius $K + \delta + k'$ in Y .

If $|F_1| \geq |F|/2$, then $|F| \leq 2N_1$ and the k -neighborhood of F in Y has at least $4N_1 \geq 2|F|$ vertices. Suppose now that $|F_1| < |F|/2$, so that $|F_2| \geq |F|/2$. Then

$$F_2 = \{Hg_1, \dots, Hg_t\}$$

where $|F_2| = t$ and where each $g_i \in G$ is shortest in Hg_i with $|g_i|_A > K + \delta + k'$. By Lemma 5.1 $(g_i, H)_1 \leq K$. By Lemma 5.2 for any $f \in G$ with $|f|_A \leq k'$ and for each $i = 1, \dots, t$ we have $(g_i f, H)_1 \leq K + \delta$.

Let $S := \{g_1, \dots, g_t\}$ and let S' be the set of all vertices of X contained in the k' -neighborhood of S in X . By the choice of k' we have $|S'| \geq 4N|S| = 4N|F_2|$. On the other hand, Lemma 5.3 implies that if $g, g' \in S'$ are such that $Hg = Hg'$ then $hg = g'$ for some $h \in H$ with $|h|_A \leq 2K + 2\delta$. By the choice of N this means that the set $F' := \{Hg \mid g \in S'\}$ contains at least

$$|S'|/N = 4N|F_2|/N = 4|F_2| \geq 2|F|$$

distinct elements. However, F' is obviously contained in the k -neighborhood of F in Y .

We have verified that for any finite nonempty subset $F \subseteq VY$ the k -neighborhood of F in Y contains at least $2|F|$ vertices. By the Doubling Condition (part 3 of Proposition 2.3) this implies that Y is nonamenable.

We can now obtain Corollary 1.4 stated in the Introduction.

COROLLARY 5.4. *Let $G = \langle x_1, \dots, x_k \mid r_1, \dots, r_m \rangle$ be a nonelementary word-hyperbolic group and let $H \leq G$ be a quasiconvex subgroup of infinite index. Let a_n be the number of freely reduced words in $A = \{x_1, \dots, x_k\}^{\pm 1}$ of length n that represent elements of H . Let b_n be the number of all words in A of length n that represent elements of H . Then*

$$\limsup_{n \rightarrow \infty} \sqrt[n]{a_n} < 2k - 1$$

and

$$\limsup_{n \rightarrow \infty} \sqrt[n]{b_n} < 2k.$$

Proof. Note that $k \geq 2$ since G is nonelementary. Put $A = \{x_1, \dots, x_k\}$ and $Y = \Gamma(G, H, A)$. We choose $x_0 := H1 \in VY$ as the base-vertex of Y . Note that Y is $2k$ -regular by construction. Also, for any vertex x of Y and any word w in $A \cup A^{-1}$ there is a unique path in Y with label w and origin x . The definition of Schreier coset graphs also implies that a word w represents an element of H if and only if the unique path in Y with origin x_0 and label w terminates at x_0 . Therefore $a_n(Y)$ equals the number of freely reduced words in the alphabet $A = \{x_1, \dots, x_k\}^{\pm 1}$ of length n that represent elements of H . Similarly, $b_n(Y)$ equals the number of all words in A of length n representing elements of H . By Theorem 1.2, Y is nonamenable. Hence by Theorem 2.5, $\alpha(Y) < 2k - 1$ and $\beta(Y) < 2k$, as required.

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