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## 4. QUASICONVEX SUBGROUPS OF HYPERBOLIC GROUPS

Detailed background information on quasiconvex subgroups of hyperbolic groups can be found in [1, 4, 20, 31, 38, 34, 32, 51, 54, 68] and other sources.

CONVENTION 4.1. Suppose  $G$  is a finitely generated group with a fixed finite generating set  $A$ . Let  $X = \Gamma(G, A)$  be the Cayley graph of  $G$  with respect to  $A$ . We will denote the word-metric corresponding to  $A$  on  $X$  by  $d_A$ . Also, for  $g \in G$  we will denote  $|g|_A := d_A(1, g)$ . For a word  $w$  in the alphabet  $A \cup A^{-1}$  we will denote by  $\bar{w}$  the element of  $G$  represented by  $w$ .

DEFINITION 4.2 (Quasiconvexity). For  $\epsilon \geq 0$  a subset  $Z$  of a metric space  $(X, d)$  is  $\epsilon$ -*quasiconvex* if, for any  $z_1, z_2 \in Z$  and any geodesic  $[z_1, z_2]$  in  $X$ , the segment  $[z_1, z_2]$  is contained in the closed  $\epsilon$ -neighborhood of  $Z$ . A subset  $Z \subseteq X$  is *quasiconvex* if it is  $\epsilon$ -quasiconvex for some  $\epsilon \geq 0$ .

If  $G$  is a finitely generated group and  $A$  is a finite generating set of  $G$ , a subgroup  $H \leq G$  is *quasiconvex in  $G$  with respect to  $A$*  if  $H \subseteq \Gamma(G, A)$  is a quasiconvex subset.

It turns out [20, 32, 4, 31] that for subgroups of word-hyperbolic groups quasiconvexity is independent of the choice of a finite generating set for the ambient group. Thus a subgroup  $H$  of a hyperbolic group  $G$  is termed *quasiconvex* if  $H \subseteq \Gamma(G, A)$  is quasiconvex for some finite generating set  $A$  of  $G$ .

We summarize some well-known basic facts regarding quasiconvex subgroups and provide some sample references:

PROPOSITION 4.3. *Let  $G$  be a word-hyperbolic group with a finite generating set  $A$ . Let  $X = \Gamma(G, A)$  be the Cayley graph of  $G$  with the word-metric  $d_A$  induced by  $A$ . Then:*

1. *If  $H \leq G$  is a subgroup, then either  $H$  is virtually cyclic (in which case  $H$  is called elementary) or  $H$  contains a free subgroup  $F$  of rank two which is quasiconvex in  $G$  (in this case  $H$  is said to be nonelementary) [20, 32].*
2. *Every cyclic subgroup of  $G$  is quasiconvex in  $G$  [1, 20, 32].*
3. *If  $H \leq G$  is quasiconvex then  $H$  is finitely presentable and word-hyperbolic [1, 20, 32].*

4. Suppose  $H \leq G$  is generated by a finite set  $Q$  inducing the word-metric  $d_Q$  on  $H$ . Then  $H$  is quasiconvex in  $G$  if and only if there is a  $C > 0$  such that for any  $h_1, h_2 \in H$

$$d_Q(h_1, h_2) \leq Cd_A(h_1, h_2)$$

(see [20, 32, 4, 31]).

5. The set  $\mathcal{L}$  of all  $A$ -geodesic words is a regular language that provides a bi-automatic structure for  $G$ . Moreover, a subgroup  $H \leq G$  is quasiconvex if and only if  $H$  is  $\mathcal{L}$ -rational, that is the set  $\mathcal{L}_H = \{w \in \mathcal{L} \mid \bar{w} \in H\}$  is a regular language [31].
6. If  $H_1, H_2 \leq G$  are quasiconvex, then  $H_1 \cap H_2 \leq G$  is quasiconvex [68].
7. [51, 46] Let  $C \leq B \leq G$  where  $B$  is quasiconvex in  $G$  (and hence  $B$  is hyperbolic) and  $C$  is quasiconvex in  $B$ . Then  $C$  is quasiconvex in  $G$  [51, 46].
8. Let  $C \leq B \leq G$  where  $C$  is quasiconvex in  $G$  and where  $B$  is word-hyperbolic. Then  $C$  is quasiconvex in  $B$  [51, 46].
9. Suppose  $H \leq G$  is an infinite quasiconvex subgroup. Then  $H$  has finite index in its commensurator  $\text{Comm}_G(H)$  (see [51]), where  $\text{Comm}_G(H) := \{g \in G \mid [H : g^{-1}Hg \cap H] < \infty \text{ and } [g^{-1}Hg : g^{-1}Hg \cap H] < \infty\}$ .

Part 1 of the above proposition implies that a nonelementary subgroup of a hyperbolic group is nonamenable.

## 5. PROOF OF THE MAIN RESULT

Let  $G$  be a nonelementary word-hyperbolic group with a finite generating set  $A$ . Let  $X = \Gamma(G, A)$  be the Cayley graph of  $G$  with the word metric  $d_A$ . Let  $\delta \geq 1$  be an integer such that the space  $(\Gamma(G, A), d_A)$  is  $\delta$ -hyperbolic. Let  $H \leq G$  be a quasiconvex subgroup of infinite index in  $G$ . These conventions, unless specified otherwise, will be fixed for the remainder of the paper.

We shall need the following useful fact:

LEMMA 5.1. *There exists an integer constant  $K = K(G, H, A) > 0$  with the following properties.*

*Assume  $g \in G$  is shortest with respect to  $d_A$  in the coset class  $Hg$ . Then for any  $h \in H$  we have  $(g, h)_1 \leq K$  (and hence  $(g, H)_1 \leq K$ ).*