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Artikel: THE NONAMENABILITY OF SCHREIER GRAPHS FOR INFINITE INDEX QUASICONVEX SUBGROUPS OF HYPERBOLIC GROUPS

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4. Quasiconvex subgroups of hyperbolic groups

Detailed background information on quasiconvex subgroups of hyperbolic groups can be found in [1, 4, 20, 31, 38, 34, 32, 51, 54, 68] and other sources.

CONVENTION 4.1. Suppose G is a finitely generated group with a fixed finite generating set A. Let $X = \Gamma(G,A)$ be the Cayley graph of G with respect to A. We will denote the word-metric corresponding to A on X by d_A. Also, for $g \in G$ we will denote $|g|_A := d_A(1,g)$. For a word w in the alphabet $A \cup A^{-1}$ we will denote by $\overline{w}$ the element of G represented by w.

DEFINITION 4.2 (Quasiconvexity). For $\epsilon \geq 0$ a subset Z of a metric space (X,d) is *ϵ-quasiconvex* if, for any $z_1, z_2 \in Z$ and any geodesic $[z_1, z_2]$ in X, the segment $[z_1, z_2]$ is contained in the closed ϵ-neighborhood of Z. A subset $Z \subseteq X$ is *quasiconvex* if it is ϵ-quasiconvex for some $\epsilon \geq 0$.

If G is a finitely generated group and A is a finite generating set of G, a subgroup $H \leq G$ is *quasiconvex in G with respect to A* if $H \subseteq \Gamma(G,A)$ is a quasiconvex subset.

It turns out [20, 32, 4, 31] that for subgroups of word-hyperbolic groups quasiconvexity is independent of the choice of a finite generating set for the ambient group. Thus a subgroup H of a hyperbolic group G is termed *quasiconvex* if $H \subseteq \Gamma(G,A)$ is quasiconvex for some finite generating set A of G.

We summarize some well-known basic facts regarding quasiconvex subgroups and provide some sample references:

PROPOSITION 4.3. *Let G be a word-hyperbolic group with a finite generating set A. Let $X = \Gamma(G,A)$ be the Cayley graph of G with the word-metric d_A induced by A. Then:*

1. *If $H \leq G$ is a subgroup, then either H is virtually cyclic (in which case H is called* elementary*) or H contains a free subgroup F of rank two which is quasiconvex in G (in this case H is said to be* nonelementary*)* [20, 32].
2. *Every cyclic subgroup of G is quasiconvex in G* [1, 20, 32].
3. *If $H \leq G$ is quasiconvex then H is finitely presentable and word-hyperbolic* [1, 20, 32].

4. *Suppose* $H \leq G$ *is generated by a finite set* Q *inducing the word-metric* d_Q *on* H. *Then* H *is quasiconvex in* G *if and only if there is a* $C > 0$ *such that for any* $h_1, h_2 \in H$

$$d_Q(h_1, h_2) \leq C d_A(h_1, h_2)$$

(*see* [20, 32, 4, 31]).
5. *The set* $\mathcal{L}$ *of all* A*-geodesic words is a regular language that provides a bi-automatic structure for* G. *Moreover, a subgroup* $H \leq G$ *is quasiconvex if and only if* H *is* $\mathcal{L}$*-rational, that is the set* $\mathcal{L}_H = \{w \in \mathcal{L} \mid \overline{w} \in H\}$ *is a regular language* [31].
6. *If* $H_1, H_2 \leq G$ *are quasiconvex, then* $H_1 \cap H_2 \leq G$ *is quasiconvex* [68].
7. [51, 46] *Let* $C \leq B \leq G$ *where* B *is quasiconvex in* G *(and hence* B *is hyperbolic) and* C *is quasiconvex in* B. *Then* C *is quasiconvex in* G [51, 46].
8. *Let* $C \leq B \leq G$ *where* C *is quasiconvex in* G *and where* B *is word-hyperbolic. Then* C *is quasiconvex in* B [51, 46].
9. *Suppose* $H \leq G$ *is an infinite quasiconvex subgroup. Then* H *has finite index in its* commensurator $Comm_G(H)$ (*see* [51]), *where* $Comm_G(H) := \{g \in G \mid [H : g^{-1}Hg \cap H] < \infty$ *and* $[g^{-1}Hg : g^{-1}Hg \cap H] < \infty\}$.

Part 1 of the above proposition implies that a nonelementary subgroup of a hyperbolic group is nonamenable.

5. Proof of the main result

Let G be a nonelementary word-hyperbolic group with a finite generating set A. Let $X = \Gamma(G, A)$ be the Cayley graph of G with the word metric d_A. Let $\delta \geq 1$ be an integer such that the space $(\Gamma(G, A), d_A)$ is δ-hyperbolic. Let $H \leq G$ be a quasiconvex subgroup of infinite index in G. These conventions, unless specified otherwise, will be fixed for the remainder of the paper.

We shall need the following useful fact:

LEMMA 5.1. *There exists an integer constant* $K = K(G, H, A) > 0$ *with the following properties.*

Assume $g \in G$ *is shortest with respect to* d_A *in the coset class* Hg. *Then for any* $h \in H$ *we have* $(g, h)_1 \leq K$ *(and hence* $(g, H)_1 \leq K$*).*