Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 48 (2002)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE NONAMENABILITY OF SCHREIER GRAPHS FOR INFINITE INDEX QUASICONVEX SUBGROUPS OF HYPERBOLIC GROUPS
Autor: Kapovich, Ilya
Kapitel: 2. Nonamenability for graphs
DOI: https://doi.org/10.5169/seals-66081

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 12.12.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

examples of such subgroups are also possible. For instance, if F is a free group of finite rank and $\phi\colon F \to F$ is an atoroidal automorphism, then the mapping torus group of ϕ

$$M_\phi = \langle F, t \mid t^{-1}ft = \phi(f) \text{ for all } f \in F \rangle$$

is word-hyperbolic [8, 13]. In this case $M_\phi/F \simeq \mathbf{Z}$ and hence the Schreier graph for M_ϕ relative to F is amenable.

The author is grateful to Laurent Bartholdi, Philip Bowers, Christophe Pittet and Tatiana Smirnova-Nagnibeda for many helpful discussions regarding random walks, to Pierre de la Harpe and Peter Brinkmann for their careful reading of the paper and numerous valuable suggestions and to Paul Schupp for encouragement.

2. Nonamenability for graphs

Let X be a connected graph of bounded degree. We define the *spectral radius* $\rho(X)$ of X as

$$\rho(X) := \limsup_{n\to\infty} \sqrt[n]{p^{(n)}(x,y)}$$

where x, y are two vertices of X and $p^{(n)}(x,y)$ is the probability that an n-step simple random walk starting at x will end up at y. It is well-known that $\rho(X) \leq 1$ and that the definition of $\rho(X)$ does not depend on the choice of x, y.

Definition 2.1 (Amenability for graphs). A connected graph X of bounded degree is said to be *amenable* if $\rho(X) = 1$ and *nonamenable* if $\rho(X) < 1$.

It is also well-known that nonamenability of X implies that X is *transient*, that is that for a simple random walk on X the probability of ever returning to the basepoint is less than 1 (see for example Theorem 51 of [16]). We refer the reader to [16, 71, 72] for comprehensive background information about random walks on graphs and for further references on this topic.

Convention 2.2. Let X be a connected graph of bounded degree with the simplicial metric d. For a finite nonempty subset $S \subset VX$ we will denote by $|S|$ the number of elements in S.

If S is a finite subset of the vertex set of X and $k \geq 1$ is an integer, we will denote by $\mathcal{N}_k^X(S) = \mathcal{N}_k(S)$ the set of all vertices v of X such that $d(v, S) \leq k$. Also, we will denote $\partial^X S = \partial S := \mathcal{N}_1(S) - S$.

The number

$$\iota(X) := \inf\{\frac{|\partial S|}{|S|} \mid S \text{ is a finite nonempty subset of the vertex set of } X\}$$

is called the *Cheeger constant* or the *isoperimetric constant* of X.

There are many alternative definitions of nonamenability:

PROPOSITION 2.3. *Let X be a connected graph of bounded degree with the simplicial metric d. Then the following conditions are equivalent:*

1. *The graph X is nonamenable.*
2. (Følner Criterion) *We have $\iota(X) > 0$.*
3. (Gromov's Doubling Condition) *There is some $k \geq 1$ such that for any finite nonempty subset $S \subseteq VX$ we have*
$$|\mathcal{N}_k(S)| \geq 2|S|.$$
4. *For any integer $q > 1$ there is some $k \geq 1$ such that for any finite nonempty subset $S \subseteq VX$ we have*
$$|\mathcal{N}_k(S)| \geq q|S|.$$
5. *For some $0 < \sigma < 1$ we have $p^{(n)}(x, y) = o(\sigma^n)$ for any $x, y \in VX$.*
6. *Let $W(X)$ be the pseudogroup of "bounded perturbations of the identity", that is $W(X)$ consists of all bijections ϕ between subsets of VX such that*
$$\sup_{x \in dom(\phi)} d(x, \phi(x)) < \infty.$$
Then $W(X)$ admits a "paradoxical decomposition", that is there exist nonempty subsets Y_1, Y_2 of VX and $\phi_1 : Y_1 \to VX$, $\phi_2 : Y_2 \to VX$ such that $\phi_1, \phi_2 \in W(X)$, $VX = Y_1 \sqcup Y_2$ and $\phi_1(Y_1) = \phi_1(Y_2) = VX$.
7. ("Grasshopper Criterion") *There exists a map $\phi : VX \to VX$ such that*
$$\sup_{x \in VX} d(x, \phi(x)) < \infty$$
and such that for any $x \in VX$ we have $\left|\phi^{-1}(x)\right| \geq 2$.
8. *There exists a map $\phi : VX \to VX$ such that*
$$\sup_{x \in VX} d(x, \phi(x)) < \infty$$
and such that for any $x \in VX$ we have $\left|\phi^{-1}(x)\right| = 2$.

9. *The bottom of the spectrum for the combinatorial Laplacian operator on* X *is* > 0 *(see* [21] *for the precise definitions).*
10. *We have* $H_0^{uf}(X) = 0$ *(see* [9] *for the precise definition of the* uniformly finite *homology groups* H_i^{uf}*).*
11. *We have* $H_0^{(l_p)}(X) = 0$ *for any* $1 < p < \infty$ *(see* [24] *for the precise definition of* $H_i^{(l_p)}$*).*

All of the above statements are well-known, but we will still provide some sample references. The fact that (1), (2), (5) and (6) are equivalent is stated in Theorem 51 of [16]. The fact that (3), (4), (6), (7) and (8) are equivalent follows from Theorem 32 of [16]. The equivalence of (2) and (9) is due to J. Dodziuk [21]. J. Block and S. Weinberger [9] established the equivalence of (2) and (10). Finally, G. Elek [24] proved that (2) is equivalent to (11).

One can characterize amenability of regular graphs in terms of cogrowth.

DEFINITION 2.4. Let X be a connected graph of bounded degree with a base-vertex x_0. Let $a_n = a_n(X, x_0)$ be the number of reduced edge-paths of length n from x_0 to x_0. Let $b_n = b_n(X, x_0)$ be the number of all edge-paths of length n from x_0 to x_0. Set

$$\alpha(X) := \limsup_{n\to\infty} \sqrt[n]{a_n} \quad \text{and} \quad \beta(X) := \limsup_{n\to\infty} \sqrt[n]{b_n}\,.$$

Then we will call $\alpha(X)$ the *cogrowth rate* of X and we will call $\beta(X)$ the *non-reduced cogrowth rate* of X. These definitions are independent of the choice of x_0.

It is easy to see that for a d-regular connected graph X we have $\alpha(X) \le d - 1$ and $\beta(X) \le d$. Moreover, $\rho(X) = \frac{\beta(X)}{d}$. The following result was originally proved by R. Grigorchuk [39] and J. Cohen [19] for the Cayley graphs of finitely generated groups and by L. Bartholdi [5] for arbitrary regular graphs.

THEOREM 2.5 ([5]). *Let* X *be a connected* d*-regular graph with* $d \ge 3$. *Set* $\alpha = \alpha(X)$, $\beta = \beta(X)$ *and* $\rho = \rho(X)$. *Then*

$$\rho = \begin{cases} \frac{2\sqrt{d-1}}{d} & \text{if } 1 \le \alpha \le \sqrt{d-1} \\ \frac{\sqrt{d-1}}{d}\left(\frac{\sqrt{d-1}}{\alpha} + \frac{\alpha}{\sqrt{d-1}}\right) & \text{if } \sqrt{d-1} \le \alpha \le d-1\,. \end{cases}$$

In particular $\rho < 1 \iff \alpha < d - 1 \iff \beta < d$.