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examples of such subgroups are also possible. For instance, if  $F$  is a free group of finite rank and  $\phi: F \rightarrow F$  is an atoroidal automorphism, then the mapping torus group of  $\phi$

$$M_\phi = \langle F, t \mid t^{-1}ft = \phi(f) \text{ for all } f \in F \rangle$$

is word-hyperbolic [8, 13]. In this case  $M_\phi/F \simeq \mathbf{Z}$  and hence the Schreier graph for  $M_\phi$  relative to  $F$  is amenable.

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## 2. NONAMENABILITY FOR GRAPHS

Let  $X$  be a connected graph of bounded degree. We define the *spectral radius*  $\rho(X)$  of  $X$  as

$$\rho(X) := \limsup_{n \rightarrow \infty} \sqrt[n]{p^{(n)}(x, y)}$$

where  $x, y$  are two vertices of  $X$  and  $p^{(n)}(x, y)$  is the probability that an  $n$ -step simple random walk starting at  $x$  will end up at  $y$ . It is well-known that  $\rho(X) \leq 1$  and that the definition of  $\rho(X)$  does not depend on the choice of  $x, y$ .

**DEFINITION 2.1** (Amenability for graphs). A connected graph  $X$  of bounded degree is said to be *amenable* if  $\rho(X) = 1$  and *nonamenable* if  $\rho(X) < 1$ .

It is also well-known that nonamenability of  $X$  implies that  $X$  is *transient*, that is that for a simple random walk on  $X$  the probability of ever returning to the basepoint is less than 1 (see for example Theorem 51 of [16]). We refer the reader to [16, 71, 72] for comprehensive background information about random walks on graphs and for further references on this topic.

**CONVENTION 2.2.** Let  $X$  be a connected graph of bounded degree with the simplicial metric  $d$ . For a finite nonempty subset  $S \subset VX$  we will denote by  $|S|$  the number of elements in  $S$ .

If  $S$  is a finite subset of the vertex set of  $X$  and  $k \geq 1$  is an integer, we will denote by  $\mathcal{N}_k^X(S) = \mathcal{N}_k(S)$  the set of all vertices  $v$  of  $X$  such that  $d(v, S) \leq k$ . Also, we will denote  $\partial^X S = \partial S := \mathcal{N}_1(S) - S$ .

The number

$$\iota(X) := \inf \left\{ \frac{|\partial S|}{|S|} \mid S \text{ is a finite nonempty subset of the vertex set of } X \right\}$$

is called the *Cheeger constant* or the *isoperimetric constant* of  $X$ .

There are many alternative definitions of nonamenability:

PROPOSITION 2.3. *Let  $X$  be a connected graph of bounded degree with the simplicial metric  $d$ . Then the following conditions are equivalent:*

1. *The graph  $X$  is nonamenable.*
2. (Følner Criterion) *We have  $\iota(X) > 0$ .*
3. (Gromov's Doubling Condition) *There is some  $k \geq 1$  such that for any finite nonempty subset  $S \subseteq VX$  we have*

$$|\mathcal{N}_k(S)| \geq 2|S|.$$

4. *For any integer  $q > 1$  there is some  $k \geq 1$  such that for any finite nonempty subset  $S \subseteq VX$  we have*

$$|\mathcal{N}_k(S)| \geq q|S|.$$

5. *For some  $0 < \sigma < 1$  we have  $p^{(n)}(x, y) = o(\sigma^n)$  for any  $x, y \in VX$ .*
6. *Let  $W(X)$  be the pseudogroup of "bounded perturbations of the identity", that is  $W(X)$  consists of all bijections  $\phi$  between subsets of  $VX$  such that*

$$\sup_{x \in \text{dom}(\phi)} d(x, \phi(x)) < \infty.$$

*Then  $W(X)$  admits a "paradoxical decomposition", that is there exist nonempty subsets  $Y_1, Y_2$  of  $VX$  and  $\phi_1: Y_1 \rightarrow VX$ ,  $\phi_2: Y_2 \rightarrow VX$  such that  $\phi_1, \phi_2 \in W(X)$ ,  $VX = Y_1 \sqcup Y_2$  and  $\phi_1(Y_1) = \phi_2(Y_2) = VX$ .*

7. ("Grasshopper Criterion") *There exists a map  $\phi: VX \rightarrow VX$  such that*

$$\sup_{x \in VX} d(x, \phi(x)) < \infty$$

*and such that for any  $x \in VX$  we have  $|\phi^{-1}(x)| \geq 2$ .*

8. *There exists a map  $\phi: VX \rightarrow VX$  such that*

$$\sup_{x \in VX} d(x, \phi(x)) < \infty$$

*and such that for any  $x \in VX$  we have  $|\phi^{-1}(x)| = 2$ .*

9. The bottom of the spectrum for the combinatorial Laplacian operator on  $X$  is  $> 0$  (see [21] for the precise definitions).
10. We have  $H_0^{uf}(X) = 0$  (see [9] for the precise definition of the uniformly finite homology groups  $H_i^{uf}$ ).
11. We have  $H_0^{(l_p)}(X) = 0$  for any  $1 < p < \infty$  (see [24] for the precise definition of  $H_i^{(l_p)}$ ).

All of the above statements are well-known, but we will still provide some sample references. The fact that (1), (2), (5) and (6) are equivalent is stated in Theorem 51 of [16]. The fact that (3), (4), (6), (7) and (8) are equivalent follows from Theorem 32 of [16]. The equivalence of (2) and (9) is due to J. Dodziuk [21]. J. Block and S. Weinberger [9] established the equivalence of (2) and (10). Finally, G. Elek [24] proved that (2) is equivalent to (11).

One can characterize amenability of regular graphs in terms of cogrowth.

DEFINITION 2.4. Let  $X$  be a connected graph of bounded degree with a base-vertex  $x_0$ . Let  $a_n = a_n(X, x_0)$  be the number of reduced edge-paths of length  $n$  from  $x_0$  to  $x_0$ . Let  $b_n = b_n(X, x_0)$  be the number of all edge-paths of length  $n$  from  $x_0$  to  $x_0$ . Set

$$\alpha(X) := \limsup_{n \rightarrow \infty} \sqrt[n]{a_n} \quad \text{and} \quad \beta(X) := \limsup_{n \rightarrow \infty} \sqrt[n]{b_n}.$$

Then we will call  $\alpha(X)$  the *cogrowth rate* of  $X$  and we will call  $\beta(X)$  the *non-reduced cogrowth rate* of  $X$ . These definitions are independent of the choice of  $x_0$ .

It is easy to see that for a  $d$ -regular connected graph  $X$  we have  $\alpha(X) \leq d - 1$  and  $\beta(X) \leq d$ . Moreover,  $\rho(X) = \frac{\beta(X)}{d}$ . The following result was originally proved by R. Grigorchuk [39] and J. Cohen [19] for the Cayley graphs of finitely generated groups and by L. Bartholdi [5] for arbitrary regular graphs.

THEOREM 2.5 ([5]). Let  $X$  be a connected  $d$ -regular graph with  $d \geq 3$ . Set  $\alpha = \alpha(X)$ ,  $\beta = \beta(X)$  and  $\rho = \rho(X)$ . Then

$$\rho = \begin{cases} \frac{2\sqrt{d-1}}{d} & \text{if } 1 \leq \alpha \leq \sqrt{d-1} \\ \frac{\sqrt{d-1}}{d} \left( \frac{\sqrt{d-1}}{\alpha} + \frac{\alpha}{\sqrt{d-1}} \right) & \text{if } \sqrt{d-1} \leq \alpha \leq d-1. \end{cases}$$

In particular  $\rho < 1 \iff \alpha < d-1 \iff \beta < d$ .