5. LORENTZIAN HOLONOMY AND HOMOGENEOUS SUBMANIFOLDS OF H^n

In this section we show how the theory of homogeneous submanifolds of the hyperbolic space H^n can be used to obtain general results about the action of a connected Lie subgroup of $\mathrm{O}(n,1)$ on the Lorentzian space $\mathbf{R}^{n,1}$, namely,

THEOREM 5.1 ([DO]). *Let G be a connected (not necessarily closed) Lie subgroup of* $\mathrm{SO}(n,1)$ *and assume that the action of G on the Lorentzian space $\mathbf{R}^{n,1}$ is weakly irreducible. Then either G acts transitively on H^n or G acts transitively on a horosphere of hyperbolic space. Moreover, if G acts irreducibly, then* $G = \mathrm{SO}_0(n,1)$.

We will explain later the concept of weak irreducibility, and we will also sketch the proof of the above theorem. First, we observe that Theorem 5.1 has an immediate corollary, which provides a purely geometric answer to a question posed in [BI].

COROLLARY 5.1 (M. Berger [B1], [B2]). *Let M be a Lorentzian manifold of dimension n. If the restricted holonomy group acts irreducibly on TM it coincides with* $\mathrm{SO}_0(n,1)$. *In particular, if M is locally symmetric then it has constant sectional curvature.*

Before giving the ideas of the proof of Theorem 5.1, we recall some basic facts on hyperbolic geometry.

Let $(V, \langle , \rangle)$ be a (real) vector space endowed with a nondegenerate symmetric bilinear form of signature $(n,1)$. It is standard to identify V with the Lorentzian space $\mathbf{R}^{n,1}$ and $Aut(\langle , \rangle) \cong \mathrm{O}(n,1)$. It is well known that the hyperbolic space H^n can be identified with a connected component of the set of points $p \in \mathbf{R}^{n,1}$ such that $\langle p, p\rangle = -1$. As in the case of the sphere, the distance $d = d(p,q)$ between two points of H^n can be computed by the equation: $\cosh(d) = -\langle p, q\rangle$. This equation comes from the fact that geodesics have the form $\exp(tv_p) = \cosh(\|v_p\|t)p + \sinh(\|v_p\|t)\frac{v_p}{\|v_p\|}$. Observe that a connected subgroup of $\mathrm{O}(n,1)$ acts on H^n by isometries. An affine subspace $q + V$ of $\mathbf{R}^{n,1}$ is called *Euclidean*, *Lorentzian* or *degenerate*, depending on whether the restriction of $\langle , \rangle$ to V is positive definite, indefinite or degenerate. A *horosphere* is a submanifold of the hyperbolic space which is obtained by intersecting H^n with an affine degenerate hyperplane. Thus, a degenerate hyperplane $q + V$ produces a foliation of H^n by parallel horospheres. The infinity $H^n(\infty)$ is the set of equivalence classes of asymptotic geodesics. It is not difficult to see that two geodesics $\exp(t.v_p)$ and $\exp(t.v_{p'})$ are asymptotic if and only if $\frac{v_p}{\|v_p\|} + p = \lambda(\frac{v'_p}{\|v'_p\|} + p')$ for some real number λ. As a consequence, we can identify the infinity $H^n(\infty)$ with the set of degenerate hyperplanes $\{\frac{v_p}{\|v_p\|} + p\}^\perp$. In this way a point z at infinity defines a foliation of H^n by parallel horospheres and we say that the horosphere Q is centred at $z \in H^n(\infty)$ if Q is a leaf of that foliation. An action of a subgroup G of $\mathrm{O}(n,1)$ is called *weakly irreducible* if it leaves invariant only degenerate subspaces.

A fundamental tool in the proof of Theorem 5.1 is the following result.

THEOREM 5.2 ([DO]). *Let G be a connected (not necessarily closed) Lie subgroup of isometries of the hyperbolic space H^n. Then one of the following assertions holds:*

(i) *G has a fixed point.*

(ii) *G has a unique non trivial totally geodesic orbit (possibly the full space).*

(iii) *All orbits are included in horospheres centred at the same point at infinity.*

The following fact plays an important role in the proof of Theorem 5.2: if a connected (not necessarily closed) Lie subgroup of isometries of the hyperbolic space H^n has a totally geodesic orbit (maybe a fixed point) then no other orbit can be minimal [DO]. A simple consequence of this fact and Theorem 5.2 is the following

THEOREM 5.3 ([DO]). *A minimal (extrinsically) homogeneous submanifold of the hyperbolic space must be totally geodesic.*

The same fact is also true in Euclidean space [D] (see also [O4]). On the other hand, it is well-known that there exist many non totally geodesic minimal (extrinsically) homogeneous submanifolds in spheres [H], [HL]. Also, there exist non totally geodesic minimal (extrinsically) homogeneous submanifolds in non compact symmetric spaces [Br]. It is interesting to note that a subgroup G of isometries of the Euclidean space always has a totally geodesic orbit (possibly a fixed point or the whole space).

A key fact in the proof of Theorem 5.2 is the following observation: if a normal subgroup H of a group G of isometries of H^n has a totally geodesic orbit $H \cdot p$ of positive dimension, then $G \cdot p = H \cdot p$. This is because G permutes H-orbits and then one can use the fact that there is a unique totally geodesic orbit, to conclude that $H \cdot p = G \cdot p$.

The next step in proving Theorem 5.2 is to study separately the following two cases: G is semisimple (of noncompact type) and G is not semisimple. In this last case one first proves the theorem for abelian groups. The above observation, applied to a normal abelian subgroup of G, implies that either G must translate a geodesic or G fixes a point at infinity or G admits a proper totally geodesic invariant submanifold. It follows that a connected Lie subgroup G of $O(n, 1)$ which acts irreducibly on $\mathbf{R}^{n,1}$ must be semisimple.

When G is a semisimple Lie group we use an Iwasawa decomposition $G = NAK$. Then one proves that the proper (solvable) subgroup NA of G has a minimal orbit which is also a G-orbit. For this, choose a fixed point p of the compact group K (which always exists by a well known theorem of Cartan). It is possible to prove that the isotropy subgroup G_p of G at p agrees with K. Then the mean curvature vector H of the orbit $G \cdot p = NA \cdot p$ is invariant by the isotropy subgroup at p and, if it is not equal to zero, the G-orbits through points on normal K-invariant geodesics turn out to be homothetical to the orbit $G \cdot p$. Observe that these orbits are also NA orbits. Finally, one can control the volume element of these orbits in terms of Jacobi fields and prove that there exists a minimal G-orbit which is also a NA-orbit.

Finally, one shows that if G has a fixed point z at infinity then either G has a totally geodesic orbit (possibly G acts transitively) or it has fixed points in H^n or all orbits of G are contained in horospheres centred at the same point z at infinity. This is because, when G has neither fixed points in H^n nor orbits in horospheres, one can construct a codimension one normal subgroup N of G such that all N-orbits are contained in the horosphere foliation defined by z. Then N acts on horospheres by isometries and one uses the fact that N must have a totally geodesic orbit in each horosphere (because each horosphere is an Euclidean space). Finally, it is not hard to show that the union of all these totally geodesic orbits over all horospheres is a totally geodesic G-invariant submanifold of H^n. Now an induction argument, involving the dimension of the Lie group G and the dimension of the corresponding hyperbolic space H^n, completes the proof of Theorem 5.2.

The proof of Theorem 5.1 runs as follows: Assume that G does not act transitively in H^n. Then G-orbits must be contained in horospheres. But if an orbit is a proper submanifold of one horosphere, one can construct a proper totally geodesic G-invariant submanifold as the union of the parallel orbits to totally geodesic orbits of the action of G, restricted to the horosphere. Then one obtains a contradiction because totally geodesic submanifolds are obtained by intersecting the hyperbolic space H^n with Lorentzian subspaces. Thus, G must act transitively on each horosphere.

Finally, if G acts irreducibly then G must act transitively on the hyperbolic space and must be semisimple of noncompact type by a previous observation. Then, showing that the isotropy group at some point agrees with a maximal compact subgroup, the second part of the theorem follows from the theory of Riemannian symmetric spaces of noncompact type [He].

REFERENCES

[AK] ALEKSEEVSKIĬ, D. V. and B. N. KIMEL′FELD. Structure of homogeneous Riemannian spaces with zero Ricci curvature (Russian). *Funkcional. Anal. i Priložen.*; English translation: *Funct. Anal. Appl. 9* (1975), no. 2, 97–102.

[BB] BÉRARD BERGERY, L. Sur la courbure des métriques riemanniennes invariantes des groupes de Lie et des espaces homogènes. *Ann. Sci. École Norm. Sup. (4) 11* (1978), 543–576.

[BI] BÉRARD BERGERY, L. and A. IKEMAKHEN. *On the Holonomy of Lorentzian Manifolds.* Proceedings of Symposia in Pure Mathematics, Volume 54 (1993), Part 2.

[B1] BERGER, M. Sur les groupes d'holonomie des variétés à connexion affine et des variétés riemanniennes. *Bull. Soc. Math. France 83* (1955), 279–330.

[B2] —— Les espaces symétriques non compacts. *Ann. Sci. École Norm. Sup. 74* (1957), 85–177.

[Br] BERNDT, J. Homogeneous hypersurfaces in hyperbolic spaces. *Math. Z. 229* (1998), 589–600.

[Be] BESSE, A. *Einstein Manifolds.* Springer, 1987.

[CW] CARTER, S and A. WEST. Partial tubes about immersed manifolds. *Geom. Dedicata 54* (1995), 145–169.

[CO] CONSOLE, S. and C. OLMOS. Submanifolds of higher rank. *Quart. J. Math. Oxford Ser. (2) 48* (1997), 309–321.

[Da] DADOK, J. Polar coordinates induced by actions of compact Lie groups. *Trans. Amer. Math. Soc. 288* (1985), 125–137.

[D] DI SCALA, A. J. Minimal homogeneous submanifolds in Euclidean spaces. *Ann. Global Anal. Geom. 21* (2002), 15–18.

[D2] —— Reducibility of complex submanifolds of the Euclidean space. *Math. Z. 235* (2000), 251–257.

[DO] DI SCALA, A. J. and C. OLMOS. The geometry of homogeneous submanifolds of hyperbolic space. *Math. Z. 237* (2001), 199–209.

[EO] ESCHENBURG, J.-H. and C. OLMOS. Rank and symmetry of Riemannian manifolds. *Comment. Math. Helv. 69* (1994), 483–499.