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Artikel: PROJECTIVE MODULES OVER SOME PRÜFER RINGS
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The sufficient conditions we state below for the strong $1\frac{1}{2}$-generator property are both already in the literature. However, the definitions we adopt in this paper are a bit different from those used in earlier work. For this reason, it will be prudent in recalling these known results to give a brief explanation for each.

Generalizing another definition introduced so far for domains (see, e.g. [FS: p. 97]), we say that a ring R has *finite character* if every *regular* element of R lies in at most finitely many maximal ideals of R; that is, for any regular element $a \in R$, R/Ra is a semilocal ring.

PROPOSITION 7 (Gilmer-Heinzer [GH]). *Any ring R of finite character has the strong $1\frac{1}{2}$ generator property.*

In fact, let $I \subseteq R$ be any invertible ideal, and let a be any given regular element in I. By assumption, a lies only in finitely many maximal ideals of R. Thus, by Theorem 3 of [GH], there exists $b \in I$ such that $I = Ra + Rb$.

PROPOSITION 8 (Griffin [Gr]). *Let R be a ring in which every regular ideal is invertible. Then R is a Prüfer ring of finite character (and hence R has the strong $1\frac{1}{2}$ generator property by Proposition 7).*

The rings in questions are, of course, exactly those Prüfer rings R whose regular ideals are f. g. (or equivalently, satisfy the ACC). By Griffin's Theorem 17 in [Gr], any regular element in such a ring R lies in only finitely many prime ideals of R; in particular, R has finite character, and so Proposition 7 applies. Examples of (commutative) rings satisfying the hypothesis of Proposition 8 include: hereditary rings, local rings whose maximal ideals consist of 0-divisors, and classical rings of quotients (e.g. 0-dimensional rings, such as von Neumann regular rings or perfect rings).

5. NON FINITELY GENERATED PROJECTIVE MODULES

In this section, we turn our attention to possibly non f. g. projective modules, and study the structure of such modules over a Prüfer ring R, assuming again that R has small 0-divisors. The goal of the section will be to prove Theorem B stated in the Introduction. We start by proving the first part of that theorem.

THEOREM 9. *Let R be a Prüfer ring with small 0-divisors, and let P be any nonzero projective R-module. Then P has a direct summand isomorphic to an invertible ideal of R. In particular, P is indecomposable if and only if it is isomorphic to an invertible ideal.*

Proof. The proof here is a more sophisticated version of that of Theorem 3. The beginning step of the argument is still the second paragraph of that proof, which works for any nonzero projective module P. In that step, we showed that, starting with any element $a \in P \setminus \operatorname{rad}(P)$, there exists a linear functional $\pi_j \colon P \to R$ with $\pi_j(a)$ regular in R. Here, we can no longer say that the ideal $\pi_j(P)$ is f.g.; however, we can proceed alternatively as follows. Following Bass [Ba: §4], let

$$o_P(a) = \{f(a) \in R : f \in P^* = \operatorname{Hom}_R(P, R)\} ,$$

and

$$o'_P(a) = \{p \in P : f(a) = 0 \Rightarrow f(p) = 0 \ \forall f \in P^*\} .$$

By [Ba: Prop. 4.1], $o'_P(a) \cong o_P(a)^*$, and $o_P(a)$ is a f.g. ideal in R. By what we said above, $o_P(a)$ contains a regular element $\pi_j(a)$. Since R is a Prüfer ring, $o_P(a)$ is an invertible ideal, and hence a projective R-module. According to Bass [Ba: Prop. 4.1], this implies that $o'_P(a)$ is a direct summand of P. Since

$$o'_P(a) \cong o_P(a)^* \cong o_P(a)^{-1} ,$$

P has a direct summand isomorphic to an invertible ideal $I \cong o_P(a)^{-1}$. And, if P is indecomposable, then $P \cong I$. □

In the following, we shall write R^∞ for the countably infinite direct sum $R \oplus R \oplus \cdots$ (as an R-module). To use this module effectively, let us recall the famous Eilenberg-Mazur trick in the following special form.

LEMMA 10. *For any ring R, we have $P \oplus R^\infty \cong R^\infty$ for any countably generated projective R-module P.*

Proof. For such a projective module P, there exists a surjection $\pi \colon R^\infty \to P$. Since π must split, we have $R^\infty \cong Q \oplus P$ for $Q = \ker(\pi)$. Thus, we have

$$\begin{aligned} P \oplus R^\infty &\cong P \oplus R \oplus R \oplus \cdots \\ &\cong P \oplus R^\infty \oplus R^\infty \oplus \cdots \\ &\cong P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \cdots \\ &\cong (P \oplus Q) \oplus (P \oplus Q) \oplus \cdots \\ &\cong R^\infty \oplus R^\infty \oplus \cdots \cong R^\infty , \end{aligned}$$

as desired. □

LEMMA 11. *Let R be a Prüfer ring with the $1\frac{1}{2}$ generator property having small 0-divisors. If $P = P_1 \oplus P_2 \oplus \cdots$ where each P_i is a nonzero countably generated projective R-module, then P is free.*

Proof. By Theorem 9, we can write $P_i \cong I_i \oplus Q_i$, where I_i is an invertible ideal in R. By the General Steinitz Isomorphism Theorem in §2, we have $I_{2i-1} \oplus I_{2i} \cong R \oplus I'_i$, where $I'_i := I_{2i-1} I_{2i} \subseteq R$. Thus,

$$\begin{aligned} P &\cong (I_1 \oplus Q_1) \oplus (I_2 \oplus Q_2) \oplus \cdots \\ &\cong [(I_1 \oplus I_2) \oplus (I_3 \oplus I_4) \oplus \cdots] \oplus (Q_1 \oplus Q_2 \oplus \cdots) \\ &\cong [(R \oplus I'_1) \oplus (R \oplus I'_2) \oplus \cdots] \oplus (Q_1 \oplus Q_2 \oplus \cdots) \\ &\cong R^\infty \oplus P' , \end{aligned}$$

where $P' := (I'_1 \oplus I'_2 \oplus \cdots) \oplus (Q_1 \oplus Q_2 \oplus \cdots)$. Since P' is a countably generated projective module, we conclude from Lemma 10 that $P \cong R^\infty \oplus P' \cong R^\infty$, as desired. □

We are now in a position to prove the rest of Theorem B.

THEOREM 12. *Let R be as in Lemma 11. Then any infinite direct sum of nonzero countably generated projective R-modules is free, and any non countably generated projective R-module is free.*

Proof. Let $P = \bigoplus_i P_i$ where the P_i's are nonzero countably generated projective R-modules, and i ranges over some *infinite* indexing set Λ. Let Λ' be another copy of Λ. Since Λ' is infinite, we have

$$\mathrm{Card}\,(\mathbf{N} \times \Lambda') = \mathrm{Card}\,(\Lambda') = \mathrm{Card}\,(\Lambda) .$$

Thus, after "identifying" Λ with $\mathbf{N} \times \Lambda'$, we can express the elements $i \in \Lambda$ in the form (n, i'), where $n \in \mathbf{N}$ and $i' \in \Lambda'$. We then have

$$P = \bigoplus_{i \in \Lambda} P_i = \bigoplus_{i' \in \Lambda'} P(i') , \tag{13}$$

where, for each $i' \in \Lambda'$, $P(i') := \bigoplus_{n \in \mathbf{N}} P_{(n,i')}$. By Lemma 11, each $P(i')$ is free, so by (13), P is also free. This proves the first part of the theorem.

For the second part, let P be any *non countably generated* projective R-module. By Kaplansky's theorem in [Ka$_3$], we can express P in the form $\bigoplus_{i \in \Lambda} P_i$ (for some indexing set Λ), where the P_i's are nonzero countably generated projective R-modules. Since P itself is not countably generated, Λ must be an infinite set. Thus, the first part of the theorem applies, showing that P is free. $\square$

It seems plausible that, under the assumptions on R in Theorem 12, any countably but not finitely generated projective R-module P is also free. This would follow from Lemma 11 if we can decompose P as in that Lemma. However, we are not able to prove the existence of such a decomposition.

We close by recalling that most results in this note required the small 0-divisor assumption on R. The study of projective modules over general Prüfer rings (without the small 0-divisor assumption) awaits further effort.

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