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## 5. TORSION NUMBERS AND LINKS

A *link* is a finite collection  $l = l_1 \cup \dots \cup l_\mu$  of pairwise disjoint knots in the 3-sphere. If a direction is chosen for each component  $l_i$ , then the link is *oriented*. Equivalence for links, possibly oriented, is defined just as for knots.

The abelianization of the group  $G = \pi_1(S^3 - l)$  is free abelian of rank  $\mu$  with generators  $t_1, \dots, t_\mu$  corresponding to oriented loops having linking number one with corresponding components of  $l$ . When  $\mu > 1$  there are infinitely many possible epimorphisms from  $G$  to the integers.

When  $l$  is oriented there is a natural choice for  $\chi$ , sending each generator  $t_i$  to  $1 \in \mathbf{Z}$ . In this way we associate to  $l$  an augmented group  $(G, \chi)$ . As in the special case of a knot,  $\mathcal{M}$  has a square presentation matrix, and it is isomorphic to the first homology group of the infinite cyclic cover of  $S^3 - l$  corresponding to  $\chi$ . Again as in the case of a knot, there is a sequence of  $r$ -fold cyclic covers  $M_r$  of  $S^3$  branched over  $l$ . However,  $H_1(M_r; \mathbf{Z})$  is isomorphic to  $\mathcal{M}/(t^{r-1} + \dots + t + 1)\mathcal{M}$  rather than  $\mathcal{M}/(t^r - 1)\mathcal{M}$  (see [Sa79]). In the case of a knot the two modules are well known to be isomorphic (see Remark 5.4(i)).

Motivated by these observations we make the following definitions. Let  $\widetilde{\mathcal{M}}_r$  denote the quotient module  $\mathcal{M}/\nu_r \mathcal{M}$ , where  $\nu_r = t^{r-1} + \dots + t + 1$ .

**DEFINITION 5.1.** Let  $(G, \chi)$  be an augmented group. The  $r^{\text{th}}$  *reduced torsion number*  $\tilde{b}_r$  is the order of the torsion submodule  $T\widetilde{\mathcal{M}}_r$ . The  $r^{\text{th}}$  *reduced Betti number*  $\tilde{\beta}_r$  is the rank of  $\widetilde{\mathcal{M}}_r$ .

As before, we may also speak of the reduced torsion and Betti numbers of a finitely generated  $\mathcal{R}_1$ -module  $\mathcal{M}$ .

Many results of Section 2 apply to reduced torsion and Betti numbers with only slight modification. For example, an argument similar to the proof of Proposition 2.1 shows that  $\tilde{\beta}_r$  is the number of zeros of the Alexander polynomial which are roots of unity and different from 1, each zero counted as many times as it occurs in the elementary divisors  $\Delta_i/\Delta_{i+1}$ ; hence  $\tilde{\beta}_r$  is periodic in  $r$ . Also, when  $\tilde{\beta}_r = 0$  the reduced torsion number  $\tilde{b}_r$  is equal to the absolute value of the resultant of  $\Delta$  and  $\nu_r$ .

**LEMMA 5.2.** Assume that  $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$  is an exact sequence of finitely generated abelian groups. If  $A$  is finite, then the induced sequence

$$0 \rightarrow A \xrightarrow{f} TB \xrightarrow{g} TC \rightarrow 0$$

is also exact.

*Proof.* The only thing to check is surjectivity of  $g$ . Since the alternating sum of the ranks of  $A, B$  and  $C$  is zero and  $A$  is finite, the ranks of  $B$  and  $C$  are equal. By Lemma 2.3 the homomorphism  $g$  maps  $TB$  onto  $TC$ .  $\square$

**PROPOSITION 5.3.** *Assume that the finitely generated  $\mathcal{R}_1$ -module  $\mathcal{M}$  has a square presentation matrix. If  $\Delta(1) \neq 0$ , then for every  $r$ ,*

$$(5.1) \quad \tilde{\beta}_r = \beta_r, \quad \tilde{b}_r = \frac{b_r}{\delta_r},$$

where  $\delta_r$  is a divisor of  $|\Delta(1)|$ . Moreover,  $\delta_{r+\gamma} = \delta_r$ , for all  $r$ , where  $\gamma$  is the cyclotomic order of  $\Delta$ .

*Proof.* Consider the sequence

$$\mathcal{M}_1 \xrightarrow{\nu_r} \mathcal{M}_r \xrightarrow{\pi} \widetilde{\mathcal{M}}_r \rightarrow 0,$$

where  $\nu_r$  is multiplication by  $\nu_r = t^{r-1} + \cdots + t + 1$ , and  $\pi$  is the natural projection. It is easy to see that the sequence is exact. From it we obtain the short exact sequence

$$0 \rightarrow \mathcal{M}_1 / \ker \nu_r \xrightarrow{\nu_r} \mathcal{M}_r \xrightarrow{\pi} \widetilde{\mathcal{M}}_r \rightarrow 0.$$

Here  $\nu_r$  also denotes the induced quotient homomorphism. Since  $\Delta(1) \neq 0$ , the module  $\mathcal{M}_1$  is finite and hence  $\beta_r = \tilde{\beta}_r$ . The order of  $\mathcal{M}_1$  is  $|\Delta(1)|$ , and hence the order of  $\mathcal{M}_1 / \ker \nu_r$  is a divisor  $\delta_r$ . The second statement of (5.1) follows from Lemmas 5.2 and 3.7.

It remains to show that  $\delta_r$  has period  $\gamma$ . For this let  $0 \neq a \in \mathcal{M}$ . The coset  $\bar{a} \in \mathcal{M}_1$  is in the kernel of  $\nu_r$  if and only if there exists  $b \in \mathcal{M}$  such that  $\nu_r(a - (t-1)b) = 0$ . Clearly this is true if and only if  $\nu_{(\gamma, r)}(a - (t-1)b) = 0$ , where  $(\gamma, r)$  denotes the gcd of  $\gamma$  and  $r$ . Hence the kernel of  $\nu_r$  is equal to the kernel of  $\nu_{(\gamma, r)}$ , and the periodicity of  $\delta_r$  follows.  $\square$

#### REMARKS 5.4.

(i) If  $G$  is a knot group, then any two meridional generators are conjugate. Consequently  $\mathcal{M}_1$  is trivial. Proposition 5.3 implies that in this case, the torsion numbers  $b_r$  and  $\tilde{b}_r$  are equal for every  $r$ .

(ii) It is well known that for any oriented link  $l = l_1 \cup l_2$  of two components,  $|\Delta(1)|$  is equal to the absolute value of the linking number  $\text{Lk}(l_1, l_2)$ . (See Theorem 7.3.16 of [Ka96].)

PROPOSITION 5.5. *Let  $\mathcal{M}$  be a finitely generated  $\mathcal{R}_1$ -module with a square presentation matrix. Assume that  $\Delta(t) = (t-1)^q g(t)$ , with  $g(1) \neq 0$ . If  $p$  is a prime that does not divide  $g(1)$ , then*

$$\tilde{\beta}_{p^k} = 0, \quad \tilde{b}_{p^k}^{(p)} = p^{qk},$$

for every  $k \geq 1$ .

The proof of Proposition 5.5 requires :

LEMMA 5.6. *Let  $g(t)$  be a polynomial with integer coefficients, and assume that  $p$  is a prime. If  $p$  does not divide  $g(1)$ , then  $p$  does not divide  $\text{Res}(g, t^{p^k} - 1)$  for any positive integer  $k$ .*

*Proof of Lemma 5.6.* Assume that  $p$  does not divide  $g(1)$ . Recall that  $\Phi_n(t)$  denotes the  $n^{\text{th}}$  cyclotomic polynomial. From the formula

$$\prod_{\substack{d|n \\ d>1}} \Phi_d(1) = \nu_n(1) = n,$$

we easily derive

$$\Phi_d(1) = \begin{cases} 0 & \text{if } d = 1 \\ q & \text{if } d = q^k > 1, q \text{ prime} \\ 1 & \text{other } d. \end{cases}$$

Consequently,  $\Phi_{p^k}$  does not divide  $g$  for any  $k > 0$ , and so  $\text{Res}(g, t^{p^k} - 1) \neq 0$ . The module  $\mathcal{H} = \mathcal{R}_1/(g, t^{p^k} - 1)$  has order  $|\text{Res}(g, t^{p^k} - 1)|$ , and it suffices to prove that  $\mathcal{H} \otimes_{\mathbb{Z}} \mathbb{Z}/p$  is trivial. Now,  $\mathcal{H} \otimes_{\mathbb{Z}} \mathbb{Z}/p$  is isomorphic to the quotient of the PID  $(\mathbb{Z}/p)[t, t^{-1}]$  by the ideal generated by the greatest common divisor of  $g$  and  $t^{p^k} - 1$  in this ring. But  $t^{p^k} - 1 = (t-1)^{p^k}$  in this ring, and  $t-1$  does not divide  $g$  since  $p$  does not divide  $g(1)$ . So the gcd is 1, and  $\mathcal{H} \otimes_{\mathbb{Z}} \mathbb{Z}/p$  is trivial.  $\square$

*Proof of Proposition 5.5.* Let  $k$  be any positive integer. Lemma 5.6 implies that  $\text{Res}(g, t^{p^k} - 1) \neq 0$ . Hence  $\beta_{p^k}$  vanishes, and therefore  $\tilde{\beta}_{p^k}$  is also zero. By a result analogous to Proposition 2.5 and the multiplicative property of resultants

$$\tilde{b}_{p^k} = |\text{Res}(\Delta, \nu_{p^k})| = |\text{Res}(t-1, \nu_{p^k})|^q |\text{Res}(g, \nu_{p^k})| = (p^k)^q |\text{Res}(g, \nu_{p^k})|.$$

By Lemma 5.6,  $p$  does not divide  $|\text{Res}(g, t^{p^k} - 1)|$ . Hence  $p$  does not divide  $\text{Res}(g, \nu_{p^k})$ , and so  $\tilde{b}_{p^k}^{(p)} = p^{kq}$ .  $\square$

COROLLARY 5.7. (i) *Let  $M_r$  be the  $r$ -fold cyclic cover of  $S^3$  branched over a knot. If  $r$  is a prime power  $p^k$ , then the  $p$ -torsion submodule of  $H_1(M_r; \mathbf{Z})$  is trivial.*

(ii) *Let  $M_r$  be the  $r$ -fold cyclic cover  $S^3$  branched over a 2-component link  $l = l_1 \cup l_2$ . If  $r$  is a power of a prime that does not divide  $\text{Lk}(l_1, l_2)$ , then the  $p$ -torsion submodule of  $H_1(M_r; \mathbf{Z})$  is trivial.*

*Proof.* Statement (i) was proven in [Go78]. Here it follows from Proposition 5.5 together with the well-known fact that  $|\Delta(1)| = 1$ , whenever  $\Delta$  is the Alexander polynomial of a knot. The second statement is a consequence of Proposition 5.5 and Remark 5.4 (ii).  $\square$

PROPOSITION 5.8. *Suppose that  $\mathcal{M}$  is a finitely generated  $\mathcal{R}_1$ -module that is isomorphic to  $\mathcal{R}_1/(\Delta)$ . If  $\Delta(t) = (t-1)^q g(t)$ , where  $g(1) \neq 0$ , then for every positive integer  $r$ , there exists a positive integer  $\delta'_r$  such that*

$$\tilde{b}_r = (\delta'_r)^q \cdot |T(\mathcal{R}_1/(g, \nu_r))|.$$

Moreover,  $\delta'_{r+\gamma} = \delta'_r$ , for all  $r$ , where  $\gamma$  is the cyclotomic order of  $\Delta$ .

### REMARKS 5.9.

(i) The order  $|T(\mathcal{R}_1/(g, \nu_r))|$  can be found using Proposition 5.3 and Theorem 3.3.

(ii) When  $\mathcal{M}$  is a direct sum of cyclic modules,  $\tilde{b}_r$  can again be found by applying Proposition 5.5 to each summand. When  $\mathcal{M}$  is not a direct sum of cyclic modules but is torsion free as an abelian group, a result analogous to Theorem 3.6 can be found by replacing  $t^r - 1$  everywhere by  $\nu_r$  in the proof. As in Section 3, the torsion numbers  $\tilde{b}_r$  are then seen to satisfy a linear homogeneous recurrence relation.

*Proof of Proposition 5.8.* Consider the exact sequence

$$0 \rightarrow \ker g \rightarrow \mathcal{R}_1/((t-1)^q, \nu_r) \xrightarrow{g} \mathcal{R}_1/((t-1)^q g, \nu_r) \xrightarrow{\pi} \mathcal{R}_1/(g, \nu_r) \rightarrow 0,$$

where the first homomorphism is inclusion, the second is multiplication by  $g$ , and the third is the natural projection. The order of  $\mathcal{R}_1/((t-1)^q, \nu_r)$  is equal to  $|\text{Res}((t-1)^q, \nu_r)|$ , which is equal to  $r^q$ . The kernel of  $g$  is generated by  $\nu_r/f_r$ , where  $f_r$  is the greatest common divisor of  $g$  and  $\nu_r$ . Notice that  $f_{r+\gamma} = f_r$ , for all  $r$ . Lemmas 5.2 and 3.7 complete the proof.  $\square$

We conclude with a generalization of Corollary 5.7 (ii).

When  $(G, \chi)$  is the augmented group corresponding to a 2-component link  $l$ , the epimorphism  $\chi$  factors through  $\eta : G \rightarrow G_{ab} \cong \mathbf{Z}^2$ . For any finite-index subgroup  $\Lambda \subset \mathbf{Z}^2$  there is a  $|\mathbf{Z}^2/\Lambda|$ -fold cover of  $S^3$  branched over  $l$  corresponding to the map  $G \rightarrow \mathbf{Z}^2 \rightarrow \mathbf{Z}^2/\Lambda$ . The cover  $M_r$  is a special case corresponding to the subgroup  $\Lambda$  generated by  $t_1 - t_2, t_1^r, t_2^r$ . We denote the rank of  $H_1(M_\Lambda; \mathbf{Z})$  by  $\beta_\Lambda$  and the order  $|TH_1(M_\Lambda; \mathbf{Z})|$  by  $b_\Lambda$ .

**THEOREM 5.10.** *Let  $l = l_1 \cup l_2$  be a link in  $S^3$ . If  $p$  is a prime that does not divide  $\text{Lk}(l_1, l_2)$ , then  $\beta_\Lambda = 0$  and  $b_\Lambda^{(p)} = 1$  for any subgroup  $\Lambda \subset \mathbf{Z}^2$  of index  $p^k$ ,  $k \geq 1$ .*

*Proof.* Let  $\mathcal{M}_\eta$  be the kernel of  $\eta$ . We consider the dual  $\mathcal{M}_\eta^\wedge$ , which is a compact abelian group with a  $\mathbf{Z}^2$ -action by automorphisms induced by conjugation in  $G$  by  $t_1$  and  $t_2$ . The automorphism induced by  $\mathbf{n} \in \mathbf{Z}^2$  is denoted by  $\sigma_{\mathbf{n}}$ ; the automorphisms induced by  $(1, 0)$  and  $(0, 1)$  are abbreviated by  $\sigma_1$  and  $\sigma_2$ , respectively. The dual  $\mathcal{M}_\eta^\wedge$  can be identified with a subspace of  $\text{Fix}_\Lambda(\sigma) = \{\rho \in \mathcal{M}_\eta^\wedge : \sigma_{\mathbf{n}}\rho = \rho \text{ for all } \mathbf{n} \in \Lambda\}$ . Details can be found in [SW00].

From the elementary ideals of  $\mathcal{M}_\eta$  a sequence of 2-variable Alexander polynomials  $\Delta_i(t_1, t_2)$  is defined; when  $i = 0$ , setting  $t_1 = t_2 = t$  recovers  $\Delta(t)$ . By [Cr65],  $\Delta_0(t_1, t_2)$  annihilates  $\mathcal{M}_\eta$ . Hence  $\Delta_0(\sigma_1, \sigma_2)\rho = 0$  for all  $\rho \in \mathcal{M}_\eta^\wedge$ . Consequently, if  $\sigma_{\mathbf{n}}\rho = \rho$  for all  $\mathbf{n} \in \mathbf{Z}^2$  then  $0 = \Delta_0(\sigma_1, \sigma_2)\rho = \Delta_0(1, 1)\rho = \Delta(1)\rho$ . Recall that  $\Delta(1) = \text{Lk}(l_1, l_2)$ .

Let

$$Y = \{\rho : \mathcal{M}_\eta \rightarrow \mathbf{Z}/p : \sigma_{\mathbf{n}}\rho = \rho \text{ for all } \mathbf{n} \in \Lambda\}.$$

We identify  $\mathbf{Z}/p$  with the group of  $p^{\text{th}}$  roots of unity, so that  $Y$  is contained in  $\mathcal{M}_\eta^\wedge$ . It is a subspace of  $\text{Fix}_\Lambda(\sigma)$  invariant under the  $\mathbf{Z}^2$ -action, and it contains a subspace isomorphic to  $\mathcal{M}_\eta \otimes_{\mathbf{Z}} \mathbf{Z}/p$ . It suffices to prove that  $Y$  is trivial.

Our hypothesis that  $p$  does not divide the linking number of  $l_1$  and  $l_2$  implies that  $\Delta_0(t_1, t_2)$  is not zero. Consequently,  $Y$  is a finite  $p$ -group and so its order is a power of  $p$ . In view of the second paragraph, the hypothesis also implies that the only point fixed by the  $\mathbf{Z}^2$ -action is trivial. But

$$|Y| = \sum |\mathcal{O}_\rho| = \sum |\mathbf{Z}^d / \text{stab}(\rho)|,$$

where the sums are taken over distinct orbits  $\mathcal{O}_\rho$  and stabilizers  $\text{stab}(\rho)$ , respectively. Each stabilizer contains  $\Lambda$ , and so  $|\mathbf{Z}^d / \text{stab}(\rho)|$  is a divisor of  $p^k$  whenever  $\rho \neq 0$ . Hence  $|Y|$  is congruent to 1 mod  $p$ . Since  $|Y|$  is a power of  $p$ , the subspace  $Y$  must be trivial.  $\square$