Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 48 (2002)

Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: TORSION NUMBERS OF AUGMENTED GROUPS WITH

APPLICATIONS TO KNOTS AND LINKS

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Kapitel: 4. Prime parts of torsion numbers

DOI: https://doi.org/10.5169/seals-66079

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4. PRIME PARTS OF TORSION NUMBERS

We recall Jensen's formula, a short argument for which can be found in [Yo86].

LEMMA 4.1 [Jensen's formula]. For any complex number α ,

$$\int_0^1 \log \left| \alpha - e^{2\pi i \theta} \right| d\theta = \log \max\{1, |\alpha|\}.$$

By Lemma 4.1 the Mahler measure M(f) of a nonzero polynomial with complex coefficients can be computed as

$$\exp \int_0^1 \log |f(e^{2\pi i\theta})| d\theta$$
.

This observation motivated the definition of Mahler measure for polynomials in several variables. (See [Bo81] or [EW99], for example.)

In [EF96], [Ev99] G.R. Everest and B.Ní Fhlathúin proved a p-adic analogue of Jensen's formula, which we describe. Assume that α is an algebraic integer lying in a finite extension K of \mathbf{Q} . For every prime p there is a p-adic absolute value $|\cdot|_p$, the usual Archimedean absolute value corresponding to ∞ . We recall the definition (see [La65] for more details): If p is a prime number, then $|p^r m/n|_p = 1/p^r$, where r is an integer, and m,n are nonzero integers that are not divisible by p. By convention, $|0|_p = 0$. Each $|\cdot|_p$ extends to an absolute value $|\cdot|_v$ on K. Let Ω_v denote the smallest field which is algebraically closed and complete with respect to $|\cdot|_v$. Let \mathbf{T}_v denote the closure of the group of all roots of unity, which is in general locally compact. Note that if $p = \infty$, then $\Omega_v = \mathbf{C}$ and $\mathbf{T}_v = \mathbf{T}$. Everest and Fhlathúin define

$$M_{\mathbf{T}_v}(t-\alpha) = \exp \int_{\mathbf{T}_v} \log|t-\alpha|_v d\mu = \exp \lim_{r\to\infty} \frac{1}{r} \sum_{\zeta^r=1} \log|\zeta-\alpha|_v.$$

Here \int denotes the Shnirelman integral, given by the limit of sums at the right, where one skips over the undefined summands. The above integral exists even if $\alpha \in \mathbf{T}_v$, in which case it can be shown to be zero. Moreover, one has

(4.1)
$$\int_{\mathbf{T}_{v}} \log|t - \alpha|_{v} d\mu = \log \max\{1, |\alpha|_{v}\},$$

which Everest and Fhlathúin refer to as a p-adic analogue of Jensen's formula.

Recall that $b_r^{(p)}$ denotes the *p-component* of b_r , the largest power of p that divides b_r . The *content* of $f \in \mathbf{Z}[t]$ is the greatest common divisor of the coefficients. Using (4.1) we will prove

THEOREM 4.2. Let (G, χ) be an augmented group, and let p be a prime.

(i) If \mathcal{M} has a square matrix presentation and $\Delta(t) \neq 0$, then the sequence $\{b_{r_k}\}$ of pure torsion numbers satisfies

$$\lim_{r_k\to\infty} (b_{r_k}^{(p)})^{1/r_k} = (\text{content } \Delta)^{(p)}.$$

(ii) If \mathcal{M} is a direct sum of cyclic modules, then the sequence of all torsion numbers satisfies

$$\lim_{r \to \infty} (b_r^{(p)})^{1/r} = (\text{content } \Delta)^{(p)}.$$

(iii) If \mathcal{M} is torsion free as an abelian group, then

$$\lim_{r \to \infty} (b_r^{(p)})^{1/r} = 1.$$

EXAMPLE 4.3. For any positive integer m, consider the augmented group (G, χ) where G is the Baumslag-Solitar group $\langle x, y \mid y^m x = xy^m \rangle$ and $\chi \colon G \to \mathbf{Z}$ maps $x \mapsto 1$ and $y \mapsto 0$. One verifies that $\mathcal{M} \cong \mathcal{R}_1/(m(t-1))$. The quotient module \mathcal{M}_r is isomorphic to $\mathbf{Z}^r/A_r\mathbf{Z}^r$, where

$$A_r = \begin{pmatrix} m & 0 & 0 & 0 & \cdots & -m \\ -m & m & 0 & \cdots & & 0 \\ 0 & -m & m & 0 & \cdots & 0 \\ & \vdots & & & & \\ 0 & 0 & \cdots & -m & m \end{pmatrix}.$$

The matrix is equivalent by elementary row and column operations to the diagonal matrix

$$\begin{pmatrix} m & & & & \\ & \ddots & & & \\ & & m & \\ & & 0 \end{pmatrix}.$$

Hence $\mathcal{M}_r \cong \mathbf{Z} \oplus (\mathbf{Z}/m)^{r-1}$, and so $b_r = m^{r-1}$ for all r. Consequently,

$$\lim_{r \to \infty} (b_r^{(p)})^{1/r} = m^{(p)}.$$

The Alexander polynomial of any knot is nonzero, and its coefficients are relatively prime. Hence the following corollary is immediate from Theorem 4.2 (iii).

COROLLARY 4.4. For any knot k and prime p,

$$\lim_{r\to\infty} (b_r^{(p)})^{1/r} = 1.$$

Theorem 2.10 and Corollary 4.4 imply that whenever the Alexander polynomial of k has Mahler measure greater than 1, infinitely many distinct primes occur in the factorization of the torsion numbers b_r . In other words, the homology groups $H_1(M_r, \mathbb{Z})$ display nontrivial p-torsion for infinitely many primes p. Since the sequence $\{b_r\}$ is a division sequence, the number of prime factors of b_r is unbounded.

What about the case in which the Alexander polynomial of k has Mahler measure equal to 1? The argument of Section 5.7 of [Go72] shows that the number of prime factors remains unbounded as long as the Alexander polynomial does not divide t^M-1 for any M. If it does divide, then the torsion numbers b_r are periodic by Section 5.3 of [Go72] (see also Corollary 2.2 of [SiWi00]). Hence we obtain

COROLLARY 4.5. For any knot, either the torsion numbers b_r are periodic or else for any N > 0 there exists an r such that the factorization of b_r has at least N distinct primes.

The proof of Theorem 4.2 requires the following lemma.

LEMMA 4.6. If $f(t) = c_0 t^n + \cdots + c_{n-1} t + c_n$ is a nonzero polynomial in $\mathbf{Z}[t]$ with roots $\lambda_1 \cdots, \lambda_n$ (not necessarily distinct) in Ω_v , then

$$|c_0|_v \prod_{i=1}^n \max\{1, |\lambda_i|_v\} = |\operatorname{content} f|_v.$$

Proof. The argument that we present is found in [LW88]. Set $a_j = c_j/c_0$ for $0 \le j \le n$, so $f(t) = c_0(t^n + a_1t^{n-1} + \cdots + a_n)$. Each a_j is an elementary symmetric function of the roots λ_i , namely the sum of products of roots taken j at a time. Using the ultrametric property

$$|x+y|_v = \max\{|x|_v, |y|_v\},$$

we see that if exactly k values of $|\lambda_i|_v$ are greater than 1, then

$$\max_{j} |a_{j}|_{v} = |a_{k}|_{v} = \prod_{j=1}^{n} \max\{1, |\lambda_{j}|_{v}\}.$$

But

$$\max_{j} |a_{j}|_{v} = \max\left\{1, \left|\frac{c_{1}}{c_{0}}\right|_{v}, \ldots, \left|\frac{c_{n}}{c_{0}}\right|_{v}\right\} = \frac{|\text{content } f|_{v}}{|c_{0}|_{v}}.$$

Hence the lemma is proved.

Proof of Theorem 4.2. In case (i), the pure torsion number b_{r_k} is equal to $\left|\prod_{\zeta^{r_k}=1}\Delta(\zeta)\right|$. We have

$$\left|b_{r_k}\right|_v = \left|\prod_{\zeta^{r_k}=1} \Delta(\zeta)\right|_v = \left|c_0\right|_v^{r_k} \prod_{\zeta^{r_k}=1} \prod_{j=1}^n \left|\zeta - \lambda_j\right|_v$$

where c_0 is the leading coefficient of Δ and $\lambda_1, \ldots \lambda_n$ are its roots. Hence

$$|b_{r_k}|_v^{1/r_k} = |c_0|_v \prod_{\zeta^{r_k}=1}^n \prod_{j=1}^n |\zeta - \lambda_j|_v^{1/r_k}$$

$$= |c_0|_v \prod_{j=1}^n \exp\left(\frac{1}{r_k} \sum_{\zeta^{r_k}=1} \log|\zeta - \lambda_j|\right),$$

so that

$$\lim_{r_k o \infty} \left| b_{r_k}
ight|_v^{1/r_k} = \left| c_0
ight|_v \prod_{j=1}^n \exp \int_{\mathbf{T}_v} \log \left| t - \lambda_j
ight|_v \ d\mu \, ,$$

which by equation (4.1) is equal to

$$|c_0|_v \prod_{j=1}^n \max\{1, |\lambda_j|_v\}.$$

By Lemma 4.6 this is equal to $|\text{content }\Delta|_v$. But for integers n we have $n^{(p)} = |n|_v^{-1}$.

Now suppose \mathcal{M} is cyclic. As in the proof of Theorem 3.8, we let γ be the cyclotomic order of Δ and consider the subsequence of b_r with r in a fixed congruence class modulo γ . Then starting with Theorem 3.3 we may apply the argument above with Δ/Φ in place of Δ to show that the limit of $(|b_r|^{(p)})^{1/r}$ along this subsequence is the p-component of the content of Δ/Φ . But content is multiplicative and cyclotomic polynomials have content 1, so the limit along all congruence classes is (content Δ)^(p). The result is immediate for direct sums of cyclic modules.

Finally, we can extend the result when \mathcal{M} is torsion-free as an abelian group using Theorem 3.6. But for this case the content of Δ is 1.