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TORSION NUMBERS OF AUGMENTED GROUPS WITH APPLICATIONS TO KNOTS AND LINKS

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Dedicated to the memory of Arnold E. Ross

ABSTRACT. Torsion and Betti numbers for knots are special cases of more general invariants b_r and β_r, respectively, associated to a finitely generated group G and epimorphism $\chi\colon G \to \mathbf{Z}$. The sequence of Betti numbers is always periodic; under mild hypotheses about (G, χ), the sequence b_r satisfies a linear homogeneous recurrence relation with constant coefficients. Generally, b_r exhibits exponential growth rate. However, again under mild hypotheses, the p-part of b_r has trivial growth for any prime p. Applications to branched cover homology for knots and links are presented.

1. INTRODUCTION

A *knot* is a simple closed curve in the 3-sphere S^3. Knots are *equivalent* if there is an orientation-preserving homeomorphism of S^3 that carries one into the other. Equivalent knots are regarded as the same. An *invariant* is a well-defined quantity that depends only on a knot equivalence class. Two knots for which some invariant differs are necessarily distinct.

Associated to any knot k and natural number r there is a compact, oriented 3-manifold M_r, the r-fold cyclic cover of S^3 branched over k. A precise definition can be found in [Li97] or [Ro76], for example. Topological invariants of M_r are invariants of k. Two such invariants, the first Betti number β_r and the order b_r of the torsion subgroup of $H_1(M_r; \mathbf{Z})$, were first considered by J. Alexander and G. Briggs [Al28], [AB27] and by O. Zariski [Za32]. The continuing interest in these invariants is witnessed by numerous papers (e.g., [Go72], [Me80], [We80], [Ri90] and [GS91]). We call b_r the r^{th} *torsion*

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number of k. We say that b_r is *pure* if the corresponding Betti number β_r vanishes (equivalently, $H_1(M_r; \mathbf{Z})$ is a pure torsion group).

Betti numbers are known to be periodic in r, and they are relatively easy to compute (see Proposition 2.2). A useful formula for pure torsion numbers was given by R. Fox in [Fo56]. Although the proof given by Fox was insufficient, a complete argument was given by C. Weber [We80]. Weber observed that the problem of computing non-pure torsion numbers is "*... une question plus difficile*".

Torsion and Betti numbers for knots are a special case of a more general, algebraic construction that depends only on an *augmented group*, consisting of a finitely generated group G and a surjection $\chi: G \to \mathbf{Z}$. We define torsion and Betti numbers in this general context. For a large class of augmented groups, including those that correspond to knots, we provide a formula for all torsion numbers, generalizing the formula of Fox. We prove that the sequence of torsion numbers satisfies a linear recurrence relation.

Torsion numbers tend to grow quickly as their index r becomes large. F. González-Acuña and H. Short [GS91] and independently R. Riley [Ri90] proved that the sequence of pure torsion numbers of any knot k has exponential growth rate equal to the Mahler measure of the Alexander polynomial of k. We improved upon this in [SW00] by showing that the entire sequence b_r grows at this rate and generalizing the result in a natural way for links. The proofs in [SW00] use a deep result about algebraic dynamical systems due to D. Lind, K. Schmidt and T. Ward (Theorem 21.1 of [Sc95]). Here we extend such results for torsion numbers b_r associated to many augmented groups. In contrast, we prove under suitable hypotheses that for any prime number p the *p-component* of b_r (i.e., the largest power of p that divides b_r) grows subexponentially. The proof relies on a p-adic version of Jensen's formula, proven by G.R. Everest and B. Ní Fhlathúin [EF96], [Ev99]. As a corollary we strengthen a theorem of C. Gordon [Go72] by proving that for any knot the sequence of torsion numbers either is periodic or else displays infinitely many prime numbers in the factorization of its terms.

In the final section we apply our techniques to the problem of computing homology groups of branched cyclic covering spaces associated to knots and links.

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2. AUGMENTED GROUPS AND TORSION NUMBERS

Torsion numbers for knots and links arise as a special case of a general group-theoretical quantity described below. We see that many knot-theoretic results remain valid in the broader context.

Let G be a finitely generated group and $\chi\colon G \to \mathbf{Z}$ any epimorphism. The pair (G,χ) is called an *augmented group*. Two augmented groups, (G_1,χ_1) and (G_2,χ_2), are *equivalent* if there exists an isomorphism $\phi\colon G_1 \to G_2$ such that $\chi_2 \circ \phi = \chi_1$.

For any augmented group (G,χ), the abelianization of ker χ is a module $\mathcal{M}$ over the ring $\mathcal{R}_1 = \mathbf{Z}[t,t^{-1}]$ of Laurent polynomials. Since $\mathcal{R}_1$ is Noetherian, $\mathcal{M}$ is finitely generated, expressible as

$$\mathcal{M} \cong \mathcal{R}_1^N/\mathcal{A}\mathcal{R}_1^M, \tag{2.1}$$

where $\mathcal{A}$ is an $N \times M$-matrix over $\mathcal{R}_1$, for some positive integers M,N. By adjoining zero columns if needed, we can assume that $M \geq N$.

For any natural number r, we define $\mathcal{M}_r$ to be the quotient module

$$\mathcal{M}_r = \mathcal{M}/(t^r-1)\mathcal{M}.$$

It is clear that $\mathcal{M}_r$ is finitely generated as an abelian group. Hence it decomposes as

$$\mathcal{M}_r \cong \mathbf{Z}^{\beta_r} \oplus T\mathcal{M}_r,$$

where $T\mathcal{M}_r$ denotes the torsion subgroup of $\mathcal{M}_r$. We define the r^{th} *torsion number* of (G,χ) to be the order b_r of $T\mathcal{M}_r$. We say that b_r is *pure* if the Betti number β_r vanishes.

Clearly b_r and β_r depend only on the module $\mathcal{M}$, which in turn depends only on the equivalence class of (G,χ). Although our motivation is group-theoretic, we note that torsion and Betti numbers can be associated as above to any finitely generated $\mathcal{R}_1$-module $\mathcal{M}$. The difference is a matter only of perspective, for it can be easily seen that any such $\mathcal{M}$ arises from an augmented group (G,χ).

The elementary ideals E_i of $\mathcal{M}$ form a sequence of invariants of (G,χ). The ideal E_i is generated by the $(N-i)\times(N-i)$ minors of the matrix $\mathcal{A}$ of (2.1). Since $\mathcal{R}_1$ is a unique factorization domain, each E_i is contained in a unique minimal principal ideal; a generator is the i^{th} *characteristic polynomial* $\Delta_i(t)$ of (G,χ), well defined up to multiplication by units in $\mathcal{R}_1$. We are primarily interested in $\Delta_0(t)$, which we abbreviate by Δ.

An important class of augmented groups arises in knot theory. For any knot k in the 3-sphere S^3 the fundamental group $G = \pi_1(S^3 - k)$ is finitely presented and has infinite cyclic abelianization. Abelianization provides a surjection $\chi\colon G \to \mathbf{Z}$. (More precisely, there are two choices. The ambiguity, which is harmless, can be eliminated by orienting the knot.) The module $\mathcal{M}$ is isomorphic to the first homology group of the infinite cyclic cover of $S^3 - k$, and it has a presentation marix $\mathcal{A}$ that is square (that is, $M = N$). The quotient module $\mathcal{M}_r$ is isomorphic to the homology group $H_1(M_r, \mathbf{Z})$ of the r-fold cyclic cover M_r of S^3 branched over k. The 0^{th} characteristic polynomial Δ is commonly called the Alexander polynomial of k. (See [Li97] or [Ro76].)

DEFINITION 2.1. The *cyclotomic order* $\gamma = \gamma(\Delta)$ is the least common multiple of those positive integers d such that the d^{th} cyclotomic polynomial Φ_d divides Δ. If no cyclotomic polynomial divides Δ then $\gamma = 1$.

PROPOSITION 2.2 (cf. Theorem 4.2 of [Go72]). *For any augmented group (G, χ) the sequence $\{\beta_r\}$ of Betti numbers satisfies $\beta_{r+\gamma} = \beta_r$, where γ is the cyclotomic order of Δ.*

Proof. We adapt an argument of D. W. Sumners that appears in [Go72].

Since $\Pi = \mathbf{C}[t, t^{-1}]$ is a principal ideal domain, the tensor product $\mathcal{M} \otimes_{\mathbf{Z}} \mathbf{C}$ decomposes as a direct sum $\oplus_{i=1}^n \Pi/(\pi_i)$, for some elements $\pi_i \in \Pi$ such that $\pi_i \mid \pi_{i+1}$, $1 \le i < n$. (For $0 \le i < n$, the product $\pi_1 \cdots \pi_{n-i}$ is the same as Δ_i up to multiplication by units in Π.) Likewise,

$$\mathcal{M}_r \otimes_{\mathbf{Z}} \mathbf{C} \cong \oplus_{i=1}^n \Pi/(\pi_i, t^r - 1).$$

Each factor $\Pi/(\pi_i)$ can be expressed as $\oplus_j \Pi/((t-\alpha_j)^{e(\alpha_j)})$, where $e(\alpha_j)$ are positive integers, α_j ranging over the distinct roots of π_i. Since

$$((t-\alpha)^{e(\alpha)}, t^r - 1) = \begin{cases} (t-\alpha) & \text{if } \alpha^r = 1, \\ \Pi & \text{otherwise}, \end{cases}$$

we see that

$$\beta_r = \dim_{\mathbf{C}} \mathcal{M}_r \otimes_{\mathbf{Z}} \mathbf{C} = \sum_{i=1}^n l_i,$$

where l_i is the number of distinct roots of π_i that are also r^{th} roots of unity. Hence $\beta_r = \beta_{(\gamma,r)}$, and so $\beta_{r+\gamma} = \beta_{(\gamma,r+\gamma)} = \beta_{(\gamma,r)} = \beta_r$. $\square$

In view of Proposition 2.2 it is natural to consider a subsequence of torsion numbers b_{r_k} such that β_{r_k} is constant. We prove that $\{b_{r_k}\}$ is a *division sequence* in the sense that b_{r_k} divides b_{r_l} whenever r_k divides r_l.

LEMMA 2.3. *Assume that $\phi\colon \mathcal{N} \to \mathcal{N}'$ is an epimorphism of finitely generated modules over a PID. If $\mathcal{N}$ and $\mathcal{N}'$ have the same rank, then ϕ restricts to an epimorphism $\phi\colon T\mathcal{N} \to T\mathcal{N}'$ of torsion submodules.*

Proof. It is clear that ϕ induces an epimorphism $\overline{\phi}\colon \mathcal{N}/T\mathcal{N} \to \mathcal{N}'/T\mathcal{N}'$. Since $\mathcal{N}$ and $\mathcal{N}'$ have the same rank, $\overline{\phi}$ is an isomorphism. If $y \in T\mathcal{N}'$, then there exists an element $x \in \mathcal{N}$ such that $\phi(x) = y$. If $x \notin T\mathcal{N}$, then x represents a nontrivial element of the kernel of $\overline{\phi}$, a contradiction. Thus ϕ restricts to an epimorphism of torsion submodules. □

PROPOSITION 2.4. *Let (G, χ) be an augmented group. If b_{r_k} is a subsequence of torsion numbers for which the corresponding Betti numbers β_{r_k} are constant, then $\{b_{r_k}\}$ is a division sequence.*

Proof. If r divides s, then clearly there exists a surjection $\phi\colon \mathcal{M}_s \to \mathcal{M}_r$. Since $\beta_r = \beta_s$, Lemma 2.3 implies that ϕ induces a surjection of torsion submodules, and consequently b_r divides b_s. □

Given an augmented group (G, χ) such that $\mathcal{M}$ has a square matrix presentation (2.1), the pure torsion numbers b_r can be computed by the following formula familiar to knot theorists.

PROPOSITION 2.5. *Assume that (G, χ) is an augmented group such that $\mathcal{M}$ has a square matrix presentation. If b_r is a pure torsion number, then it is equal to the absolute value of*

$$\prod_{\zeta^r=1} \Delta(\zeta)\,. \tag{2.2}$$

The quantity (2.2) is equal to the resultant $\mathrm{Res}(\Delta, t^r - 1)$. In general, if $f(t) = a_0t^n + \cdots + a_{n-1}t + a_n$ and $g(t) = b_0t^m + \cdots + b_{m-1}t + b_m$ are polynomials with integer coefficients and zeros $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_m$, respectively, then the *resultant* of f and g is

$$\mathrm{Res}(f, g) = (a_0^m b_0^n) \prod_{i,j} (\alpha_i - \beta_j) = a_0^m \prod_i g(\alpha_i) = (-1)^{mn} b_0^n \prod_j f(\beta_j)\,.$$

Clearly, $\mathrm{Res}(f_1f_2, g) = \mathrm{Res}(f_1, g)\mathrm{Res}(f_2, g)$ and $\mathrm{Res}(f, g) = (-1)^{mn}\mathrm{Res}(g, f)$. The resultant has an alternative definition as the determinant of a certain matrix formed from the coefficients of f and g (cf. [La65]). In particular, the resultant of integer polynomials is always an integer.

In the case that G is a knot group, formula (2.2) was given by R. Fox [Fo56]. A complete proof is contained in [We80]. The proof of Proposition 2.5 can be fashioned along similar lines. We will prove a more general result in Section 3.

In [Le33] D. H. Lehmer investigated resultants $\mathrm{Res}(f, t^r - 1)$, where $f(t) \in \mathbf{Z}[t]$. As he observed, it follows from a theorem of Lagrange that the sequence $\{\mathrm{Res}(f, t^r - 1)\}$ satisfies a linear homogeneous recurrence relation in r with constant coefficients.

The general linear recurrence relation is easy to find. Assume that $f(t) = c_0 t^d + \cdots + c_{d-1}t + c_d$ has roots $\alpha_1, \ldots, \alpha_d$. Form the polynomials

$$\begin{aligned}
f_0(t) &= t - 1\,, \\
f_1(t) &= \frac{1}{c_0} f(t) = \prod_{i=1}^{d} (t - \alpha_i)\,, \\
f_2(t) &= \prod_{i>j=1}^{d-1} (t - \alpha_i \alpha_j)\,, \\
&\ \vdots \\
f_d(t) &= t - \alpha_1 \alpha_2 \cdots \alpha_d = t - (-1)^d \frac{c_d}{c_0}\,.
\end{aligned}$$

It is not necessary to find the roots of f in order to determine $f_0, \ldots, f_d$. The coefficients of these polynomials are integers obtained rationally in terms of the coefficients of f. Lehmer gives explicit formulas for $d < 6$ ([Le33], p. 472–3). If $t^m + A_1 t^{m-1} + \cdots + A_m$ is the least common multiple of $f_0, \ldots, f_d$, then $\mathrm{Res}(f, t^r - 1)$, which we abbreviate by $R(f, r)$, satisfies the homogeneous linear recurrence with characteristic polynomial $p(t) = c_0^m t^m + c_0^{m-1} A_1 t^{m-1} + \cdots + A_m$; that is,

$$c_0^m R(f, r+m) + c_0^{m-1} A_1 R(f, r+m-1) + \cdots + A_m R(f, r) = 0\,. \tag{2.3}$$

It is easy to see that the degree m of the characteristic equation (2.3) is not greater than 2^d. These facts were rediscovered by W. Stevens [St00]. Stevens proved that when f is a reciprocal polynomial (that is, $c_i = c_{d-i}$ for $i = 0, 1, \ldots, d$) this degree m can be bounded from above by $3^{d/2}$.

We remark that the sign of $\mathrm{Res}(f, t^r - 1)$ is either constant or alternating. For in the product

$$\mathrm{Res}(f, t^r - 1) = c_0^m \prod_i (\alpha_i^r - 1)\,,$$

a pair of conjugate complex roots contributes a factor $(\alpha_i^r - 1)(\overline{\alpha}_i^r - 1) = |\alpha_i^r - 1|^2$, while the real factors have constant or alternating sign. It follows that $|\mathrm{Res}(f, t^t - 1)|$ satisfies a linear recurrence of the same order as $\mathrm{Res}(f, t^r - 1)$; in the alternating sign case, simply modify the characteristic polynomial by changing the sign of alternate terms.

EXAMPLE 2.6. The Alexander polynomial of the figure-eight knot (the knot 4_1 in tables) is $\Delta(t) = t^2 - 3t + 1$. Since neither root has modulus one, all of the torsion numbers of k are pure. The polynomials f_i are $f_0(t) = f_2(t) = t - 1$ and $f_1(t) = \Delta(t)$. The least common multiple is $t^3 - 4t^2 + 4t - 1$, and hence b_r satisfies: $b_{r+3} - 4b_{r+2} + 4b_{r+1} - b_r = 0$. Using the initial conditions $b_0 = 0, b_1 = 1, b_2 = 5$, other values can now be quickly computed.

The torsion numbers for the figure-eight knot produce some surprisingly large prime factors. According to calculations done with Maple, b_{1361} is the square of a prime with 285 digits.

Lehmer, who considered this example in [Le33], albeit for much smaller values of r, was interested in producing new prime numbers. He observed that the factors of $R(f, r)$ satisfy a severe arithmetical constraint, and he proposed that if $R(f, r)$ grows with a relatively small exponential growth rate, then these numbers will likely display large prime factors. Lehmer did not give any proof of the assertion about prime factors, but rather used it heuristically. A survey of Lehmer's efforts together with new results in these directions can be found in [EEW00].

DEFINITION 2.7. Assume that

$$f(t) = c_0 t^d + \cdots + c_{d-1} t + c_d = c_0 \prod_{i=1}^{d} (t - \alpha_i)$$

is a polynomial with complex coefficients, $c_0 \neq 0$. The *Mahler measure* of f is

$$M(f) = |c_0| \prod_{i=1}^{d} \max\{1, |\alpha_i|\}.$$

The empty product is assumed to be 1, so that the Mahler measure of a nonzero constant polynomial $f(t) = c_0$ is $|c_0|$. By convention, the Mahler measure of the zero polynomial is zero.

Clearly, Mahler measure is multiplicative; that is, $M(fg) = M(f)M(g)$, for $f, g \in \mathbf{C}[t]$. The following is proved in [GS91] and [Ri90]. We sketch the argument.

PROPOSITION 2.8. *Let f be a polynomial with integer coefficients. The subsequence $R(f, r_k)$ of nonvanishing resultants has exponential growth rate $M(f)$; that is,*

$$\lim_{r_k \to \infty} |\mathrm{Res}(f, t^{r_k} - 1)|^{1/r_k} = M(f).$$

Sketch of proof. Let $f(t) = c_0 t^d + \cdots + c_{d-1} t + c_d$. Assume that $c_0 \neq 0$ and that $\alpha_1, \ldots, \alpha_d$ (not necessarily distinct) are the roots of f. Then

$$|\mathrm{Res}(f, t^r - 1)|^{1/r} = |c_0| \prod_{i=1}^{d} |\alpha_i^r - 1|^{1/r}.$$

The condition that the resultant does not vanish is equivalent to the statement that no root α_i is an r^{th} root of unity. Consider the subsequence of natural integers r for which this is the case. Note that if $|\alpha_i| < 1$, then the factor $|\alpha_i^r - 1|^{1/r}$ converges to 1 as r goes to infinity. On the other hand, if $|\alpha_i| > 1$, then for sufficiently large r we have

$$\frac{1}{2}|\alpha_i|^r \le |\alpha_i|^r - 1 \le |\alpha_i^r - 1| \le |\alpha_i|^r + 1 \le 2|\alpha_i|^r.$$

Taking r^{th} roots we see that $|\alpha_i^r - 1|^{1/r}$ converges to $|\alpha_i|$.

When some root α_i lies on the unit circle the nonzero values of $|\alpha_i^r - 1|$ can fluctuate wildly. In this case the analysis is more subtle. González-Acuña and Short use results of A. Baker [Ba77] and A.O. Gelfond [Ge35] to obtain estimates. In [GS91] it is shown that if $|\alpha_i^r| \neq 1$, then

$$C \exp\{-(\log r)^6\} < |\alpha_i^r - 1| \le 2,$$

where C is a positive constant that depends only on f. As in the case that $|\alpha_i| < 1$ we have that $|\alpha_i^r - 1|^{1/r}$ converges to 1.

The conclusion of Proposition 2.8 follows. □

The following is immediate from Propositions 2.8 and 2.5.

COROLLARY 2.9. *Assume that the finitely generated $\mathcal{R}_1$-module $\mathcal{M}$ has a square matrix presentation. Then the subsequence of $\{b_r\}$ consisting of pure torsion numbers has exponential growth rate equal to $M(\Delta)$.*

We can extend the conclusion of Proposition 2.8 to the entire sequence of resultants by using results from the theory of algebraic dynamical systems. Only the essential elements of the theory are sketched below. Readers unfamiliar with dynamical systems might refer to [EW99].

In brief, to a finitely generated $\mathcal{R}_1$-module we associate a compact space and a homeomorphism σ from the space to itself. The fixed points of σ^r form a closed subspace consisting of exactly b_r connected components. Topological techniques are available to compute the exponential growth rate of b_r, and it coincides with $M(\Delta)$.

THEOREM 2.10. *Assume that the finitely generated $\mathcal{R}_1$-module $\mathcal{M}$ either (i) has a square presentation matrix; or (ii) is torsion-free as an abelian group. Then the sequence $\{b_r\}$ of torsion numbers has exponential growth rate equal to $M(\Delta)$.*

Proof. Let $\mathcal{M}^\wedge$ denote the Pontryagin dual $\mathrm{Hom}(\mathcal{M}, \mathbf{T})$; that is, the topological group of homomorphisms ρ from $\mathcal{M}$ to the additive circle group $\mathbf{T} = \mathbf{R}/\mathbf{Z}$. Here $\mathcal{M}$ has the discrete topology, and $\mathcal{M}^\wedge$ the compact-open topology. Multiplication by t in $\mathcal{M}$ induces a homeomorhism σ of $\mathcal{M}^\wedge$ defined by $\sigma(\rho)(a) = \rho(ta)$, for any $\rho \in \mathcal{M}^\wedge$ and all $a \in \mathcal{M}$. The dual of $\mathcal{M}_r = \mathcal{M}/(t^r - 1)\mathcal{M}$ is the subspace $\mathrm{Fix}(\sigma^r) = \{\rho \in \mathcal{M}^\wedge \mid \sigma^r \rho = \rho\}$, the set of points of $\mathcal{M}^\wedge$ with *period* r.

Since $\mathcal{M}_r = \mathbf{Z}^{\beta_r} \oplus T\mathcal{M}_r$, the dual $\mathcal{M}_r^\wedge$ is homeomorphic to $\mathbf{T}^{\beta_r} \times T\mathcal{M}_r$. This follows from two facts: $\mathbf{Z}^\wedge$ is isomorphic to $\mathbf{T}$; and $A^\wedge$ is isomorphic to A for any finite abelian group. Hence the number of connected components of $\mathcal{M}_r^\wedge$ is equal to the cardinality of $T\mathcal{M}_r$, which by definition is the torsion number b_r. Each component is a torus of dimension β_r, a beautiful fact but one that we will not use here.

The number of connected components of $\mathcal{M}_r^\wedge$ is the same as the number N_r of connected components of $\mathrm{Fix}(\sigma^r)$. Theorem 21.1(3) of [Sc95] states that the exponential growth rate of N_r is equal to the topological entropy of σ. (The proof of this deep result uses a definition of topological entropy in terms of *separating sets*. For an elementary discussion of the theorem see [EW99].)

Further, if $\mathcal{M}$ has a presentation (2.1) with square matrix $\mathcal{A}$, then the topological entropy of σ is equal to $M(\Delta)$. (See Example 18.7(1) of [Sc95].) Thus if the hypothesis (i) is satisfied, then we are done.

If $\mathcal{M}$ is torsion-free as an abelian group, then again the topological entropy of σ is equal to $M(\Delta)$ by Lemma 17.6 of [Sc95]. □

The hypotheses of Theorem 2.10 cannot be dropped, as the following example illustrates.

EXAMPLE 2.11. Consider the augmented group (G,χ) such that

$$G = \langle x, a \mid x^{-2}a^2xa^{-6}xa^2,\ x^{-3}axa^{-4}xa^4xa^{-1}\rangle\,,$$

and $\chi\colon G \to \mathbf{Z}$ maps $x \mapsto 1$ and $a \mapsto 0$. A straightforward calculation shows that $\mathcal{M} \cong \mathcal{R}_1/(2f,(t-1)f)$, where $f(t) = t^2-3t+1$. The Alexander polynomial Δ is $\gcd(2f,(t-1)f) = f$, and it has Mahler measure greater than 1. However, the topological entropy of the homeomorphism σ is zero by Corollary 18.5 of [Sc95]. As in the proof of the theorem above, it follows that the torsion numbers b_r have trivial exponential growth rate; that is, $\limsup_{r\to\infty} b_r^{1/r} = 1$.

3. EXTENDED FOX FORMULA AND RECURRENCE

Let (G,χ) be an augmented group, and $\mathcal{A}$ the $N\times M$ presentation matrix for the $\mathcal{R}_1$-module $\mathcal{M}$ as in (2.1). For any positive integer r we can obtain a presentation matrix for the finitely generated abelian group $\mathcal{M}_r$ by replacing each entry $q(t)$ of $\mathcal{A}$ by the $r\times r$ block $q(C_r)$, where C_r is the companion matrix of t^r-1,

$$C_r = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

We call the resulting $rN\times rM$ matrix $\mathcal{A}(C_r)$. The proof is not difficult. The torsion number b_r is equal to the absolute value of the product of the nonzero elementary divisors of $\mathcal{A}(C_r)$.

Assume first that $\mathcal{M}$ is a cyclic module. Then $\mathcal{A}$ is the 1×1 matrix $(\Delta(t))$, and the $r\times r$ matrix $(\Delta(C_r))$ presents $\mathcal{M}_r$. The Betti number β_r is the number of zeros of Δ that are r^{th} roots of unity. When it vanishes the matrix $(\Delta(C_r))$ is nonsingular. Then all elementary divisors of the matrix are nonzero, and their product is equal (up to sign) to the product of the eigenvalues, which is the determinant. Fox's formula (Proposition 2.5) follows by choosing a basis for $\mathbf{C}^r$ that diagonalizes the companion matrix C_r; we then see that the eigenvalues of $\Delta(C_r)$ are $\Delta(\zeta)$, where ζ ranges over the r^{th} roots of unity. In general, β_r is equal to

$$s = \sum_{\substack{d|r \\ \Phi_d|\Delta}} \deg \Phi_d = \sum_{\substack{d|r \\ \Phi_d|\Delta}} \phi(d),$$

where Φ_d is, as before, the d^{th} cyclotomic polynomial, and ϕ is Euler's phi function. We appeal to the following result, a special case of Theorem 2.1 of [MM82].

LEMMA 3.1. *Let A be an integral $r \times r$ matrix with rank $r - s$. Suppose that R is an integral $s \times r$ matrix with an $s \times s$ minor invertible over $\mathbf{Z}$ such that $RA = 0$ and $AR^T = 0$ (where R^T denotes the transpose matrix). Then the product of the nonzero eigenvalues of A is equal to $\pm\det(RR^T)$ times the product of the nonzero elementary divisors of A.*

EXAMPLE 3.2. Suppose that we have a factorization $t^r - 1 = \Phi \cdot \Psi$ in $\mathbf{Z}[t]$. Set $A = \Phi(C_r)$. Then we can construct a matrix R satisfying the hypotheses of Lemma 3.1. We regard $\mathcal{R}_1/(t^r - 1)$ as a free abelian group with generators $1, t, \ldots, t^{r-1}$. Then the rows of A represent the polynomials $\Phi, t\Phi, \ldots, t^{r-1}\Phi$ (modulo $t^r - 1$). The rank of A is $r - s$, where $s = \deg \Phi$. We take R to be the $s \times r$ matrix with rows representing $\Psi, t\Psi, \ldots, t^{s-1}\Psi$. Consider first the product RA. Regarding the product of the i^{th} row of R with A as a linear combination of the rows of A, we see that it represents the polynomial $t^{i-1}\Psi \cdot \Phi \equiv 0$ (modulo $t^r - 1$). Hence $RA = 0$.

The columns of A represent the polynomials $\Phi(t^{-1})$, $t\Phi(t^{-1}), \ldots,$ $t^{r-1}\Phi(t^{-1})$, and so the i^{th} column of AR^T represents $\Phi(t^{-1}) \cdot t^i\Psi(t)$ (modulo $t^r - 1$). Since Φ is a product of cyclotomic polynomials, we have $t^{\deg \Phi}\Phi(t^{-1}) = \pm\Phi(t)$. (A cyclotomic polynomial has this property since its set of roots is preserved by inversion, and its leading and constant coefficients are ± 1.) So AR^T is also zero.

Since the degree of $t^i\Psi$ is less than r for $i < s$, the $s \times s$ minor consisting of the first s columns of R is upper triangular. The diagonal entries are the constant term of Ψ, which must be ± 1. Hence this minor is invertible over $\mathbf{Z}$.

The matrix $\mathcal{A}$ presents $\mathcal{R}_1/(\Phi, t^r - 1) \cong \mathcal{R}_1/(\Phi)$, a free abelian group, so the product of its elementary divisors is 1. Lemma 3.1 implies that $\det(RR^T)$

is equal up to sign to the product of the nonzero eigenvalues of $\Phi(C_r)$; that is,

$$\det(RR^T) = \pm \prod_{\substack{\zeta^r=1 \\ \Phi(\zeta)\neq 0}} \Phi(\zeta). \tag{3.1}$$

THEOREM 3.3. *Suppose that the $\mathcal{R}_1$-module $\mathcal{M}$ is isomorphic to $\mathcal{R}_1/(\Delta)$. For any positive integer r, let Φ be the product of the distinct cyclotomic polynomials Φ_d such that $d \mid r$ and $\Phi_d \mid \Delta$. Then*

$$b_r = \Big| \prod_{\substack{\zeta^r=1 \\ \Delta(\zeta)\neq 0}} \left(\frac{\Delta}{\Phi}\right)(\zeta) \Big| . \tag{3.2}$$

REMARKS 3.4.

(i) We follow the convention that if no cyclotomic polynomial divides Δ, then $\Phi = 1$. Clearly b_r is a pure torsion number if and only if $\Phi = 1$. In this case (3.2) reduces to Fox's formula (2.2).

(ii) See [Sa95] and [HS97] for more calculations and estimations of torsion numbers b_r arising from link groups.

Proof of Theorem 3.3. We write Δ as $\Phi \cdot g$, for some $g \in \mathbf{Z}[t]$. The matrix $\Delta(C_r)$, which presents $\mathcal{M}_r = \mathcal{R}_1/(\Delta, t^r - 1)$, has rank $r - \deg \Phi$. The rank is the same as that of $\Phi(C_r)$. Consider the matrix R of Example 3.2. We have $R\Delta(C_r) = (R\Phi(C_r))g(C_r) = 0$ and also $\Delta(C_r)R^T = (\Phi(C_r)g(C_r))R^T = g(C_r)(\Phi(C_r)R^T) = 0$. Formula (3.2) now follows from Lemma 3.1 together with (3.1). $\square$

If $\mathcal{M}$ is a direct sum of cyclic modules, then Theorem 3.3 can be applied to each summand and the terms produced by (3.2) multiplied together in order to compute b_r.

When $\mathcal{M}$ is not necessarily a direct sum of cyclic modules, but it is torsion-free as an abelian group, then it is "virtually" a direct sum of cyclic modules by the following lemma, which appears as Lemma 9.1 in [Sc95]. The main idea of the proof is to consider the natural injection of $\mathcal{M} \hookrightarrow \mathcal{M} \otimes_{\mathbf{Z}} \mathbf{Q}$, and use the fact that $\mathcal{M} \otimes_{\mathbf{Z}} \mathbf{Q}$ is a finitely generated module over the ring $\mathbf{Q}[t^{\pm 1}]$, which is a principal ideal domain.

We recall that a polynomial in $\mathbf{Z}[t]$ is said to be primitive if the only constants that divide it are ± 1.

LEMMA 3.5. *Assume that $\mathcal{M}$ is a finitely generated $\mathcal{R}_1$-module that is torsion-free as an abelian group. Then there exist primitive polynomials $\pi_1, \dots, \pi_n \in \mathbf{Z}[t]$ such that $\pi_i \mid \pi_{i+1}$ for all $i = 1, \dots, n-1$, and an $\mathcal{R}_1$-module injection $i\colon \mathcal{M} \to \mathcal{M}' = \mathcal{R}_1/(\pi_1) \oplus \cdots \oplus \mathcal{R}_1/(\pi_n)$ such that $\mathcal{M}'/i(\mathcal{M})$ is finite.*

For notational convenience we identify $\mathcal{M}$ with its image in $\mathcal{M}'$. Consider the mappings $\mu\colon \mathcal{M} \to \mathcal{M}$ and $\mu'\colon \mathcal{M}' \to \mathcal{M}'$ given by $a \mapsto (t^r - 1)a$. Clearly $\ker \mu$ is a submodule of $\ker \mu'$. We define $\kappa(r)$ to be the index $|\ker \mu' : \ker \mu|$. Let b_r' denote the order of the torsion subgroup of $\mathcal{M}'/(t^r - 1)\mathcal{M}'$. The proof of the following theorem extends techniques of [We80].

THEOREM 3.6. *If the finitely generated $\mathcal{R}_1$-module $\mathcal{M}$ is torsion-free as an abelian group, then for any positive integer r,*

$$b_r = \frac{b_r'}{\kappa(r)}. \tag{3.3}$$

Moreover, if γ is the cyclotomic order of Δ, then $\kappa(r+\gamma) = \kappa(r)$ for all r.

LEMMA 3.7. *Let $0 \to A_1 \to A_2 \to \cdots \to A_m \to 0$ be an exact sequence of finite abelian groups. Then*

$$\prod |A_{\text{even}}| = \prod |A_{\text{odd}}|.$$

Lemma 3.7 is easily proved using induction on m. We leave the details to the reader.

Proof of Theorem 3.6. Consider the finite quotient $p\colon \mathcal{M}' \to \mathcal{M}'/\mathcal{M}$ and mapping $\bar{\mu}\colon \mathcal{M}'/\mathcal{M} \to \mathcal{M}'/\mathcal{M}$ given by $a \mapsto (t^r - 1)a$. The exact diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{M} \arrow[r, "i"] \arrow[d, "\mu"'] & \mathcal{M}' \arrow[r, "p"] \arrow[d, "\mu'"'] & \mathcal{M}'/\mathcal{M} \arrow[r] \arrow[d, "\bar{\mu}"'] & 0 \\
0 \arrow[r] & \mathcal{M} \arrow[r, "i"] & \mathcal{M}' \arrow[r, "p"] & \mathcal{M}'/\mathcal{M} \arrow[r] & 0
\end{tikzcd}$$

induces a second exact diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker \mu \arrow[r, "i"] \arrow[d] & \ker \mu' \arrow[r, "p"] \arrow[d] & \ker \overline{\mu} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{M} \arrow[r, "i"] \arrow[d, "\mu"'] & \mathcal{M}' \arrow[r, "p"] \arrow[d, "\mu'"'] & \mathcal{M}'/\mathcal{M} \arrow[r] \arrow[d, "\overline{\mu}"'] & 0 \\
0 \arrow[r] & \mathcal{M} \arrow[r, "i"] \arrow[d] & \mathcal{M}' \arrow[r, "p"] \arrow[d] & \mathcal{M}'/\mathcal{M} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{M}_r \arrow[r, "\overline{i}"] \arrow[d] & \mathcal{M}'_r \arrow[r, "\overline{p}"] \arrow[d] & \operatorname{coker} \overline{\mu} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

and hence by the Snake Lemma we obtain a long exact sequence

(3.4) $$0 \to \ker \mu \xrightarrow{i} \ker \mu' \xrightarrow{p} \ker \overline{\mu} \xrightarrow{d} \mathcal{M}_r \xrightarrow{\overline{i}} \mathcal{M}'_r \xrightarrow{\overline{p}} \operatorname{coker} \overline{\mu} \to 0\,.$$

Let $T\mathcal{M}_r$ and $T\mathcal{M}'_r$ be the torsion subgroups of $\mathcal{M}_r$ and $\mathcal{M}'_r$, respectively. Since $\ker \overline{\mu}$ is finite, its image under the connecting homomorphism d is contained in $T\mathcal{M}_r$. Also, $\overline{i}$ maps $T\mathcal{M}_r$ into $T\mathcal{M}'_r$. Hence we have an induced sequence

(3.5) $$0 \to \ker \mu \xrightarrow{i} \ker \mu' \xrightarrow{p} \ker \overline{\mu} \xrightarrow{d} T\mathcal{M}_r \xrightarrow{\overline{i}} T\mathcal{M}'_r \xrightarrow{\overline{p}} \operatorname{coker} \overline{\mu} \to 0\,.$$

It is not difficult to verify that (3.5) is exact. The only nonobvious thing to check is that the kernel of $\overline{p}$ is contained in the image of $\overline{i}$. To see this, assume that $\overline{p}(y) = 0$. By the exactness of (3.4) there exists an element $x \in \mathcal{M}_r$ such that $\overline{i}(x) = y$. If $x \notin T\mathcal{M}_r$, then the multiples of x are distinct in $\mathcal{M}_r$ and each maps by $\overline{i}$ into the finite group $T\mathcal{M}'_r$, contradicting the fact that $\ker \overline{i} = d(\ker \overline{\mu})$ is finite.

The following sequence is exact.

(3.6) $$0 \to \ker \mu'/i(\ker \mu) \to \ker \overline{\mu} \to T\mathcal{M}_r \to T\mathcal{M}'_r \to \operatorname{coker} \overline{\mu} \to 0\,.$$

Since $\mathcal{M}'_r/\mathcal{M}_r$ is finite, $\ker \overline{\mu}$ and $\operatorname{coker} \overline{\mu}$ have the same order. Lemma 3.7 now completes the proof of (3.3), $\kappa(r)$ being the order of $\ker \mu'/i(\ker \mu)$.

The modules $\mathcal{M}$ and $\mathcal{M}'$ have characteristic polynomial π_n. Since $\mathcal{M}$ embeds in $\mathcal{M}'$ with finite index, a prime polynomial annihilates a nonzero element of $\mathcal{M}$ if and only if it annihilates a nonzero element of $\mathcal{M}'$. Such polynomials are exactly the prime divisors of π_n. It follows that $\ker \mu$ and $\ker \mu'$ are both periodic, with period equal to the least common multiple γ of the positive integers d such that Φ_d divides Δ. Hence the same is true for $\kappa(r)$. $\square$

THEOREM 3.8. *Assume that the finitely generated $\mathcal{R}_1$-module $\mathcal{M}$ is a direct sum of cyclic modules or is torsion free as an abelian group. Then the set of torsion numbers b_r satisfies a linear homogeneous recurrence relation with constant coefficients.*

Proof. Write

$$\Delta = \Big(\prod_{d\in D} \Phi_d^{e_d}\Big) \cdot g ,$$

where $D = \{d \ : \ \Phi_d \mid \Delta\}$, and let γ be the cyclotomic order of Δ. We will show that for each $R \in \{0, \dots, \gamma - 1\}$, the subsequence of b_r with r congruent to R modulo γ satisfies

$$b_r = C_R\, r^{M_R} |\mathrm{Res}(g, t^r - 1)| , \tag{3.7}$$

where C_R, M_R are constants,

$$M_R = \sum_{\substack{d\in D\\ d\mid R}} \phi(d)(e_d - 1) \le M = \sum_{d\in D} \phi(d)(e_d - 1) .$$

As we saw in section 2, the sequence $|\mathrm{Res}(g, t^r - 1)|$ satisfies a linear homogeneous recurrence relation with characteristic polynomial p of degree at most $2^{\deg g}$. We may normalize p to be monic, $p(t) = \prod_j (t - \lambda_j)^{n_j}$, with λ_j distinct. The general solution to this recurrence relation has the form $\sum_j q_j(r)\lambda_j^r$, where q_j is a polynomial of degree less than n_j (see [Br92], Theorem 7.2.2, for example). Each of the sequences $a_r^{(R)} = C_R r^{M_R} |\mathrm{Res}(g, t^r - 1)|$ satisfies the recurrence relation given by $\hat{p}(t) = \prod_j (t - \lambda_j)^{n_j + M}$. It also satisfies the recurrence relation given by $P(t) = \prod_j (t^\gamma - \lambda_j^\gamma)^{n_j + M}$, since $\hat{p}$ divides P. Because the powers of t occurring in P are all multiples of γ, the latter recurrence relation also describes the sequence $\{b_r\}$, which is composed of the subsequences $b_{R+\gamma n} = a_{R+\gamma n}^{(R)}$. We note that the degree of P is at most $\gamma(M+1)2^{\deg g}$.

First we consider the case when $\mathcal{M}$ is cyclic. Given R we set

$$\Phi = \prod_{\substack{d\in D\\ d|R}} \Phi_d .$$

By Theorem 3.3 we have

$$b_r = \Big| \prod_{\substack{\zeta^r=1\\ \Delta(\zeta)\neq 0}} \Big(\frac{\Delta}{\Phi}\Big)(\zeta)\Big| = \Big|\operatorname{Res}\Big(\frac{\Delta}{\Phi}, \frac{t^r-1}{\Phi}\Big)\Big|$$

$$= \prod_{d\in D} \Big|\operatorname{Res}\Big(\Phi_d, \frac{t^r-1}{\Phi}\Big)\Big|^{e'_d} \Big|\operatorname{Res}\Big(g, \frac{t^r-1}{\Phi}\Big)\Big| ,$$

where

$$e'_d = \begin{cases} e_d - 1 & \text{if } d \mid R, \\ e_d & \text{if } d \nmid R. \end{cases}$$

For each d dividing R,

$$\operatorname{Res}\Big(\Phi_d, \frac{t^r-1}{\Phi}\Big) = \prod_{\Phi_d(\omega)=0} \frac{t^r-1}{\Phi(t)}\Big|_{t=\omega}$$

$$= \prod_{\Phi_d(\omega)=0} \frac{(t^d-1)(1+t^d+\cdots+t^{(r/d-1)d})}{\Phi_d(t)\widehat{\Phi}(t)}\Big|_{t=\omega}$$

$$= \prod_{\Phi_d(\omega)=0} \left[\frac{t^d-1}{\Phi_d(t)}\Big|_{t=\omega} \cdot \frac{r/d}{\widehat{\Phi}(\omega)}\right] = C_d \cdot r^{\phi(d)} ,$$

where $\widehat{\Phi} = \Phi/\Phi_d$ and C_d depends only on d and R. For $d \in D$ not dividing R,

$$\operatorname{Res}(\Phi_d, \frac{t^r-1}{\Phi}) = \prod_{\Phi_d(\omega)=0} \frac{\omega^r-1}{\Phi(\omega)}$$

is constant for r congruent to R modulo γ, since d divides γ. Finally,

$$\operatorname{Res}(g, \frac{t^r-1}{\Phi}) = c_0^{r-\deg \Phi} \prod_{g(\alpha)=0} \frac{\alpha^r-1}{\Phi(\alpha)} ,$$

where c_0 is the leading coefficient of g; the expression can be rewritten as $C\operatorname{Res}(g, t^r-1)$, where C depends only on R. Thus we can express b_r in the desired form (3.7) for all r congruent to R modulo γ.

For the case when $\mathcal{M}$ is a direct sum of cyclic modules $\mathcal{R}_1/(\pi_1)\oplus\cdots\oplus\mathcal{R}_1/(\pi_n)$ we apply the above result to each summand and use the facts that $\Delta = \pi_1 \ldots \pi_n$ and b_r is the product of the torsion numbers of the summands to see that equation (3.7) still holds. Finally, if $\mathcal{M}$ is torsion free as an abelian group, we use Theorem 3.6. $\square$

4. PRIME PARTS OF TORSION NUMBERS

We recall Jensen's formula, a short argument for which can be found in [Yo86].

LEMMA 4.1 [Jensen's formula]. *For any complex number* α,

$$\int_0^1 \log\left|\alpha - e^{2\pi i\theta}\right| d\theta = \log\max\{1, |\alpha|\}.$$

By Lemma 4.1 the Mahler measure $M(f)$ of a nonzero polynomial with complex coefficients can be computed as

$$\exp \int_0^1 \log\left|f(e^{2\pi i\theta})\right| d\theta .$$

This observation motivated the definition of Mahler measure for polynomials in several variables. (See [Bo81] or [EW99], for example.)

In [EF96], [Ev99] G.R. Everest and B. Ní Fhlathúin proved a p-adic analogue of Jensen's formula, which we describe. Assume that α is an algebraic integer lying in a finite extension K of $\mathbf{Q}$. For every prime p there is a p-adic absolute value $|\cdot|_p$, the usual Archimedean absolute value corresponding to ∞. We recall the definition (see [La65] for more details): If p is a prime number, then $|p^r m/n|_p = 1/p^r$, where r is an integer, and m, n are nonzero integers that are not divisible by p. By convention, $|0|_p = 0$. Each $|\cdot|_p$ extends to an absolute value $|\cdot|_v$ on K. Let Ω_v denote the smallest field which is algebraically closed and complete with respect to $|\cdot|_v$. Let $\mathbf{T}_v$ denote the closure of the group of all roots of unity, which is in general locally compact. Note that if $p = \infty$, then $\Omega_v = \mathbf{C}$ and $\mathbf{T}_v = \mathbf{T}$. Everest and Fhlathúin define

$$M_{\mathbf{T}_v}(t-\alpha) = \exp \int_{\mathbf{T}_v} \log|t-\alpha|_v d\mu = \exp \lim_{r\to\infty} \frac{1}{r} \sum_{\zeta^r=1} \log|\zeta-\alpha|_v .$$

Here $\int$ denotes the Shnirelman integral, given by the limit of sums at the right, where one skips over the undefined summands. The above integral exists even if $\alpha \in \mathbf{T}_v$, in which case it can be shown to be zero. Moreover, one has

$$\int_{\mathbf{T}_v} \log|t-\alpha|_v d\mu = \log\max\{1, |\alpha|_v\}, \tag{4.1}$$

which Everest and Fhlathúin refer to as a p-adic analogue of Jensen's formula.

Recall that $b_r^{(p)}$ denotes the *p-component* of b_r, the largest power of p that divides b_r. The *content* of $f \in \mathbf{Z}[t]$ is the greatest common divisor of the coefficients. Using (4.1) we will prove

THEOREM 4.2. *Let (G, χ) be an augmented group, and let p be a prime.*

(i) *If $\mathcal{M}$ has a square matrix presentation and $\Delta(t) \neq 0$, then the sequence $\{b_{r_k}\}$ of pure torsion numbers satisfies*

$$\lim_{r_k \to \infty} (b_{r_k}^{(p)})^{1/r_k} = (\text{content } \Delta)^{(p)} .$$

(ii) *If $\mathcal{M}$ is a direct sum of cyclic modules, then the sequence of all torsion numbers satisfies*

$$\lim_{r \to \infty} (b_r^{(p)})^{1/r} = (\text{content } \Delta)^{(p)} .$$

(iii) *If $\mathcal{M}$ is torsion free as an abelian group, then*

$$\lim_{r \to \infty} (b_r^{(p)})^{1/r} = 1 .$$

EXAMPLE 4.3. For any positive integer m, consider the augmented group (G, χ) where G is the Baumslag-Solitar group $\langle x, y \mid y^m x = x y^m \rangle$ and $\chi \colon G \to \mathbf{Z}$ maps $x \mapsto 1$ and $y \mapsto 0$. One verifies that $\mathcal{M} \cong \mathcal{R}_1/(m(t-1))$. The quotient module $\mathcal{M}_r$ is isomorphic to $\mathbf{Z}^r/A_r\mathbf{Z}^r$, where

$$A_r = \begin{pmatrix} m & 0 & 0 & 0 & \cdots & -m \\ -m & m & 0 & \cdots & & 0 \\ 0 & -m & m & 0 & \cdots & 0 \\ & & \vdots & & & \\ 0 & 0 & & \cdots & -m & m \end{pmatrix} .$$

The matrix is equivalent by elementary row and column operations to the diagonal matrix

$$\begin{pmatrix} m & & & \\ & \ddots & & \\ & & m & \\ & & & 0 \end{pmatrix} .$$

Hence $\mathcal{M}_r \cong \mathbf{Z} \oplus (\mathbf{Z}/m)^{r-1}$, and so $b_r = m^{r-1}$ for all r. Consequently,

$$\lim_{r \to \infty} (b_r^{(p)})^{1/r} = m^{(p)} .$$

The Alexander polynomial of any knot is nonzero, and its coefficients are relatively prime. Hence the following corollary is immediate from Theorem 4.2 (iii).

COROLLARY 4.4. *For any knot k and prime p,*

$$\lim_{r\to\infty} (b_r^{(p)})^{1/r} = 1 .$$

Theorem 2.10 and Corollary 4.4 imply that whenever the Alexander polynomial of k has Mahler measure greater than 1, infinitely many distinct primes occur in the factorization of the torsion numbers b_r. In other words, the homology groups $H_1(M_r, \mathbf{Z})$ display nontrivial p-torsion for infinitely many primes p. Since the sequence $\{b_r\}$ is a division sequence, the number of prime factors of b_r is unbounded.

What about the case in which the Alexander polynomial of k has Mahler measure equal to 1 ? The argument of Section 5.7 of [Go72] shows that the number of prime factors remains unbounded as long as the Alexander polynomial does not divide $t^M - 1$ for any M. If it does divide, then the torsion numbers b_r are periodic by Section 5.3 of [Go72] (see also Corollary 2.2 of [SiWi00]). Hence we obtain

COROLLARY 4.5. *For any knot, either the torsion numbers b_r are periodic or else for any $N > 0$ there exists an r such that the factorization of b_r has at least N distinct primes.*

The proof of Theorem 4.2 requires the following lemma.

LEMMA 4.6. *If $f(t) = c_0 t^n + \cdots + c_{n-1} t + c_n$ is a nonzero polynomial in $\mathbf{Z}[t]$ with roots $\lambda_1 \cdots, \lambda_n$ (not necessarily distinct) in Ω_v, then*

$$|c_0|_v \prod_{i=1}^{n} \max\{1, |\lambda_i|_v\} = |\text{content}\, f|_v .$$

Proof. The argument that we present is found in [LW88]. Set $a_j = c_j / c_0$ for $0 \le j \le n$, so $f(t) = c_0(t^n + a_1 t^{n-1} + \cdots + a_n)$. Each a_j is an elementary symmetric function of the roots λ_i, namely the sum of products of roots taken j at a time. Using the ultrametric property

$$|x + y|_v = \max\{|x|_v, |y|_v\} ,$$

we see that if exactly k values of $|\lambda_i|_v$ are greater than 1, then

$$\max_j |a_j|_v = |a_k|_v = \prod_{j=1}^{n} \max\{1, |\lambda_j|_v\} .$$

But

$$\max_j |a_j|_v = \max\{1, \left|\frac{c_1}{c_0}\right|_v, \dots, \left|\frac{c_n}{c_0}\right|_v\} = \frac{|\text{content } f|_v}{|c_0|_v}.$$

Hence the lemma is proved. □

Proof of Theorem 4.2. In case (i), the pure torsion number b_{r_k} is equal to $\left|\prod_{\zeta^{r_k}=1} \Delta(\zeta)\right|$. We have

$$|b_{r_k}|_v = \left|\prod_{\zeta^{r_k}=1} \Delta(\zeta)\right|_v = |c_0|_v^{r_k} \prod_{\zeta^{r_k}=1} \prod_{j=1}^{n} |\zeta - \lambda_j|_v ,$$

where c_0 is the leading coefficient of Δ and $\lambda_1, \dots \lambda_n$ are its roots. Hence

$$\begin{aligned} |b_{r_k}|_v^{1/r_k} &= |c_0|_v \prod_{\zeta^{r_k}=1} \prod_{j=1}^{n} |\zeta - \lambda_j|_v^{1/r_k} \\ &= |c_0|_v \prod_{j=1}^{n} \exp\left(\frac{1}{r_k} \sum_{\zeta^{r_k}=1} \log|\zeta - \lambda_j|\right), \end{aligned}$$

so that

$$\lim_{r_k \to \infty} |b_{r_k}|_v^{1/r_k} = |c_0|_v \prod_{j=1}^{n} \exp \int_{\mathbf{T}_v} \log|t - \lambda_j|_v \, d\mu ,$$

which by equation (4.1) is equal to

$$|c_0|_v \prod_{j=1}^{n} \max\{1, |\lambda_j|_v\}.$$

By Lemma 4.6 this is equal to $|\text{content } \Delta|_v$. But for integers n we have $n^{(p)} = |n|_v^{-1}$.

Now suppose $\mathcal{M}$ is cyclic. As in the proof of Theorem 3.8, we let γ be the cyclotomic order of Δ and consider the subsequence of b_r with r in a fixed congruence class modulo γ. Then starting with Theorem 3.3 we may apply the argument above with Δ/Φ in place of Δ to show that the limit of $(|b_r|^{(p)})^{1/r}$ along this subsequence is the p-component of the content of Δ/Φ. But content is multiplicative and cyclotomic polynomials have content 1, so the limit along all congruence classes is (content $\Delta)^{(p)}$. The result is immediate for direct sums of cyclic modules.

Finally, we can extend the result when $\mathcal{M}$ is torsion-free as an abelian group using Theorem 3.6. But for this case the content of Δ is 1. □

5. TORSION NUMBERS AND LINKS

A *link* is a finite collection $l = l_1 \cup \cdots \cup l_\mu$ of pairwise disjoint knots in the 3-sphere. If a direction is chosen for each component l_i, then the link is *oriented*. Equivalence for links, possibly oriented, is defined just as for knots.

The abelianization of the group $G = \pi_1(S^3 - l)$ is free abelian of rank μ with generators $t_1, \ldots, t_\mu$ corresponding to oriented loops having linking number one with corresponding components of l. When $\mu > 1$ there are infinitely many possible epimorphisms from G to the integers.

When l is oriented there is a natural choice for χ, sending each generator t_i to $1 \in \mathbf{Z}$. In this way we associate to l an augmented group (G, χ). As in the special case of a knot, $\mathcal{M}$ has a square presentation matrix, and it is isomorphic to the first homology group of the infinite cyclic cover of $S^3 - l$ corresponding to χ. Again as in the case of a knot, there is a sequence of r-fold cyclic covers M_r of S^3 branched over l. However, $H_1(M_r; \mathbf{Z})$ is isomorphic to $\mathcal{M}/(t^{r-1} + \cdots + t + 1)\mathcal{M}$ rather than $\mathcal{M}/(t^r - 1)\mathcal{M}$ (see [Sa79]). In the case of a knot the two modules are well known to be isomorphic (see Remark 5.4(i)).

Motivated by these observations we make the following definitions. Let $\widetilde{\mathcal{M}}_r$ denote the quotient module $\mathcal{M}/\nu_r\mathcal{M}$, where $\nu_r = t^{r-1} + \cdots + t + 1$.

DEFINITION 5.1. Let (G, χ) be an augmented group. The r^{th} *reduced torsion number* $\widetilde{b}_r$ is the order of the torsion submodule $T\widetilde{\mathcal{M}}_r$. The r^{th} *reduced Betti number* $\widetilde{\beta}_r$ is the rank of $\widetilde{\mathcal{M}}$.

As before, we may also speak of the reduced torsion and Betti numbers of a finitely generated $\mathcal{R}_1$-module $\mathcal{M}$.

Many results of Section 2 apply to reduced torsion and Betti numbers with only slight modification. For example, an argument similar to the proof of Proposition 2.1 shows that $\widetilde{\beta}_r$ is the number of zeros of the Alexander polynomial which are roots of unity and different from 1, each zero counted as many times as it occurs in the elementary divisors Δ_i/Δ_{i+1}; hence $\widetilde{\beta}_r$ is periodic in r. Also, when $\widetilde{\beta}_r = 0$ the reduced torsion number $\widetilde{b}_r$ is equal to the absolute value of the resultant of Δ and ν_r.

LEMMA 5.2. *Assume that* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *is an exact sequence of finitely generated abelian groups. If* A *is finite, then the induced sequence*

$$0 \to A \xrightarrow{f} TB \xrightarrow{g} TC \to 0$$

is also exact.

Proof. The only thing to check is surjectivity of g. Since the alternating sum of the ranks of A, B and C is zero and A is finite, the ranks of B and C are equal. By Lemma 2.3 the homomorphism g maps TB onto TC. □

PROPOSITION 5.3. *Assume that the finitely generated $\mathcal{R}_1$-module $\mathcal{M}$ has a square presentation matrix. If $\Delta(1) \neq 0$, then for every r,*

$$\widetilde{\beta}_r = \beta_r\,, \qquad \widetilde{b}_r = \frac{b_r}{\delta_r}\,, \tag{5.1}$$

where δ_r is a divisor of $|\Delta(1)|$. Moreover, $\delta_{r+\gamma} = \delta_r$, for all r, where γ is the cyclotomic order of Δ.

Proof. Consider the sequence

$$\mathcal{M}_1 \xrightarrow{\nu_r} \mathcal{M}_r \xrightarrow{\pi} \widetilde{\mathcal{M}}_r \to 0\,,$$

where ν_r is multiplication by $\nu_r = t^{r-1} + \cdots + t + 1$, and π is the natural projection. It is easy to see that the sequence is exact. From it we obtain the short exact sequence

$$0 \to \mathcal{M}_1/\ker \nu_r \xrightarrow{\nu_r} \mathcal{M}_r \xrightarrow{\pi} \widetilde{\mathcal{M}}_r \to 0\,.$$

Here ν_r also denotes the induced quotient homomorphism. Since $\Delta(1) \neq 0$, the module $\mathcal{M}_1$ is finite and hence $\beta_r = \widetilde{\beta}_r$. The order of $\mathcal{M}_1$ is $|\Delta(1)|$, and hence the order of $\mathcal{M}_1/\ker \nu_r$ is a divisor δ_r. The second statement of (5.1) follows from Lemmas 5.2 and 3.7.

It remains to show that δ_r has period γ. For this let $0 \neq a \in \mathcal{M}$. The coset $\bar{a} \in \mathcal{M}_1$ is in the kernel of ν_r if and only if there exists $b \in \mathcal{M}$ such that $\nu_r(a - (t-1)b) = 0$. Clearly this is true if and only if $\nu_{(\gamma,r)}(a - (t-1)b) = 0$, where (γ, r) denotes the gcd of γ and r. Hence the kernel of ν_r is equal to the kernel of $\nu_{(\gamma,r)}$, and the periodicity of δ_r follows. □

REMARKS 5.4.

(i) If G is a knot group, then any two meridianal generators are conjugate. Consequently $\mathcal{M}_1$ is trivial. Proposition 5.3 implies that in this case, the torsion numbers b_r and $\widetilde{b}_r$ are equal for every r.

(ii) It is well known that for any oriented link $l = l_1 \cup l_2$ of two components, $|\Delta(1)|$ is equal to the absolute value of the linking number $\mathrm{Lk}(l_1, l_2)$. (See Theorem 7.3.16 of [Ka96].)

PROPOSITION 5.5. *Let $\mathcal{M}$ be a finitely generated $\mathcal{R}_1$-module with a square presentation matrix. Assume that $\Delta(t) = (t-1)^q g(t)$, with $g(1) \neq 0$. If p is a prime that does not divide $g(1)$, then*

$$\widetilde{\beta}_{p^k} = 0, \quad \widetilde{b}^{(p)}_{p^k} = p^{qk},$$

for every $k \geq 1$.

The proof of Proposition 5.5 requires:

LEMMA 5.6. *Let $g(t)$ be a polynomial with integer coefficients, and assume that p is a prime. If p does not divide $g(1)$, then p does not divide* $\mathrm{Res}(g, t^{p^k} - 1)$ *for any positive integer k.*

Proof of Lemma 5.6. Assume that p does not divide $g(1)$. Recall that $\Phi_n(t)$ denotes the n^{th} cyclotomic polynomial. From the formula

$$\prod_{\substack{d|n \\ d>1}} \Phi_d(1) = \nu_n(1) = n,$$

we easily derive

$$\Phi_d(1) = \begin{cases} 0 & \text{if } d = 1 \\ q & \text{if } d = q^k > 1,\ q \text{ prime} \\ 1 & \text{other } d. \end{cases}$$

Consequently, Φ_{p^k} does not divide g for any $k > 0$, and so $\mathrm{Res}(g, t^{p^k} - 1) \neq 0$. The module $\mathcal{H} = \mathcal{R}_1/(g, t^{p^k} - 1)$ has order $|\mathrm{Res}(g, t^{p^k} - 1)|$, and it suffices to prove that $\mathcal{H} \otimes_{\mathbf{Z}} \mathbf{Z}/p$ is trivial. Now, $\mathcal{H} \otimes_{\mathbf{Z}} \mathbf{Z}/p$ is isomorphic to the quotient of the PID $(\mathbf{Z}/p)[t, t^{-1}]$ by the ideal generated by the greatest common divisor of g and $t^{p^k} - 1$ in this ring. But $t^{p^k} - 1 = (t-1)^{p^k}$ in this ring, and $t-1$ does not divide g since p does not divide $g(1)$. So the gcd is 1, and $\mathcal{H} \otimes_{\mathbf{Z}} \mathbf{Z}/p$ is trivial. $\square$

Proof of Proposition 5.5. Let k be any positive integer. Lemma 5.6 implies that $\mathrm{Res}(g, t^{p^k} - 1) \neq 0$. Hence β_{p^k} vanishes, and therefore $\widetilde{\beta}_{p^k}$ is also zero. By a result analagous to Proposition 2.5 and the multiplicative property of resultants

$$\widetilde{b}_{p^k} = |\mathrm{Res}(\Delta, \nu_{p^k})| = |\mathrm{Res}(t-1, \nu_{p^k})|^q |\mathrm{Res}(g, \nu_{p^k})| = (p^k)^q |\mathrm{Res}(g, \nu_{p^k})|.$$

By Lemma 5.6, p does not divide $|\mathrm{Res}(g, t^{p^k} - 1)|$. Hence p does not divide $\mathrm{Res}(g, \nu_{p^k})$, and so $b^{(p)}_{p^k} = p^{kq}$. $\square$

COROLLARY 5.7. (i) *Let M_r be the r-fold cyclic cover of S^3 branched over a knot. If r is a prime power p^k, then the p-torsion submodule of $H_1(M_r;\mathbf{Z})$ is trivial.*

(ii) *Let M_r be the r-fold cyclic cover S^3 branched over a 2-component link $l = l_1 \cup l_2$. If r is a power of a prime that does not divide $\mathrm{Lk}(l_1, l_2)$, then the p-torsion submodule of $H_1(M_r;\mathbf{Z})$ is trivial.*

Proof. Statement (i) was proven in [Go78]. Here it follows from Proposition 5.5 together with the well-known fact that $|\Delta(1)| = 1$, whenever Δ is the Alexander polynomial of a knot. The second statement is a consequence of Proposition 5.5 and Remark 5.4 (ii). □

PROPOSITION 5.8. *Suppose that $\mathcal{M}$ is a finitely generated $\mathcal{R}_1$-module that is isomorphic to $\mathcal{R}_1/(\Delta)$. If $\Delta(t) = (t-1)^q g(t)$, where $g(1) \neq 0$, then for every positive integer r, there exists a positive integer δ'_r such that*

$$\widetilde{b}_r = (\delta'_r)^q \cdot |T(\mathcal{R}_1/(g, \nu_r))| .$$

Moreover, $\delta'_{r+\gamma} = \delta'_r$, for all r, where γ is the cyclotomic order of Δ.

REMARKS 5.9.

(i) The order $|T(\mathcal{R}_1/(g, \nu_r))|$ can be found using Proposition 5.3 and Theorem 3.3.

(ii) When $\mathcal{M}$ is a direct sum of cyclic modules, $\widetilde{b}_r$ can again be found by applying Proposition 5.5 to each summand. When $\mathcal{M}$ is not a direct sum of cyclic modules but is torsion free as an abelian group, a result analogous to Theorem 3.6 can be found by replacing $t^r - 1$ everywhere by ν_r in the proof. As in Section 3, the torsion numbers $\widetilde{b}_r$ are then seen to satisfy a linear homogeneous recurrence relation.

Proof of Proposition 5.8. Consider the exact sequence

$$0 \to \ker g \to \mathcal{R}_1/((t-1)^q, \nu_r) \xrightarrow{g} \mathcal{R}_1/((t-1)^q g, \nu_r) \xrightarrow{\pi} \mathcal{R}_1/(g, \nu_r) \to 0 ,$$

where the first homomorphism is inclusion, the second is multiplication by g, and the third is the natural projection. The order of $\mathcal{R}_1/((t-1)^q, \nu_r)$ is equal to $|\mathrm{Res}((t-1)^q, \nu_r)|$, which is equal to r^q. The kernel of g is generated by ν_r/f_r, where f_r is the greatest common divisor of g and ν_r. Notice that $f_{r+\gamma} = f_r$, for all r. Lemmas 5.2 and 3.7 complete the proof. □

We conclude with a generalization of Corollary 5.7 (ii).

When (G,χ) is the augmented group corresponding to a 2-component link l, the epimorphism χ factors through $\eta : G \to G_{ab} \cong \mathbf{Z}^2$. For any finite-index subgroup $\Lambda \subset \mathbf{Z}^2$ there is a $|\mathbf{Z}^2/\Lambda|$-fold cover of S^3 branched over l corresponding to the map $G \to \mathbf{Z}^2 \to \mathbf{Z}^2/\Lambda$. The cover M_r is a special case corresponding to the subgroup Λ generated by $t_1 - t_2$, t_1^r, t_2^r. We denote the rank of $H_1(M_\Lambda;\mathbf{Z})$ by β_Λ and the order $|TH_1(M_\Lambda;\mathbf{Z})|$ by b_Λ.

THEOREM 5.10. *Let $l = l_1 \cup l_2$ be a link in S^3. If p is a prime that does not divide* $\mathrm{Lk}(l_1,l_2)$, *then $\beta_\Lambda = 0$ and $b_\Lambda^{(p)} = 1$ for any subgroup $\Lambda \subset \mathbf{Z}^2$ of index p^k, $k \geq 1$.*

Proof. Let $\mathcal{M}_\eta$ be the kernel of η. We consider the dual $\mathcal{M}_\eta^\wedge$, which is a compact abelian group with a $\mathbf{Z}^2$-action by automorphisms induced by conjugation in G by t_1 and t_2. The automorphism induced by $\mathbf{n} \in \mathbf{Z}^2$ is denoted by $\sigma_\mathbf{n}$; the automorphims induced by $(1,0)$ and $(0,1)$ are abbreviated by σ_1 and σ_2, respectively. The dual $\mathcal{M}_\eta^\wedge$ can be identified with a subspace of $\mathrm{Fix}_\Lambda(\sigma) = \{\rho \in \mathcal{M}_\eta^\wedge \ : \ \sigma_\mathbf{n}\rho = \rho \text{ for all } \mathbf{n} \in \Lambda\}$. Details can be found in [SW00].

From the elementary ideals of $\mathcal{M}_\eta$ a sequence of 2-variable Alexander polynomials $\Delta_i(t_1,t_2)$ is defined; when $i = 0$, setting $t_1 = t_2 = t$ recovers $\Delta(t)$. By [Cr65], $\Delta_0(t_1,t_2)$ annihilates $\mathcal{M}_\eta$. Hence $\Delta_0(\sigma_1,\sigma_2)\rho = 0$ for all $\rho \in \mathcal{M}_\eta^\wedge$. Consequently, if $\sigma_\mathbf{n}\rho = \rho$ for all $\mathbf{n} \in \mathbf{Z}^2$ then $0 = \Delta_0(\sigma_1,\sigma_2)\rho = \Delta_0(1,1)\rho = \Delta(1)\rho$. Recall that $\Delta(1) = \mathrm{Lk}(l_1,l_2)$.

Let
$$Y = \{\rho\colon \mathcal{M}_\eta \to \mathbf{Z}/p \ : \ \sigma_\mathbf{n}\rho = \rho \text{ for all } \mathbf{n} \in \Lambda\}.$$
We identify $\mathbf{Z}/p$ with the group of p^{th} roots of unity, so that Y is contained in $\mathcal{M}_\eta^\wedge$. It is a subspace of $\mathrm{Fix}_\Lambda(\sigma)$ invariant under the $\mathbf{Z}^2$-action, and it contains a subspace isomorphic to $\mathcal{M}_\eta \otimes_\mathbf{Z} \mathbf{Z}/p$. It suffices to prove that Y is trivial.

Our hypothesis that p does not divide the linking number of l_1 and l_2 implies that $\Delta_0(t_1,t_2)$ is not zero. Consequently, Y is a finite p-group and so its order is a power of p. In view of the second paragraph, the hypothesis also implies that the only point fixed by the $\mathbf{Z}^2$-action is trivial. But
$$|Y| = \sum |\mathcal{O}_\rho| = \sum \left|\mathbf{Z}^d/\operatorname{stab}(\rho)\right|,$$
where the sums are taken over distinct orbits $\mathcal{O}_\rho$ and stabilizers $\operatorname{stab}(\rho)$, respectively. Each stabilizer contains Λ, and so $\left|\mathbf{Z}^d/\operatorname{stab}(\rho)\right|$ is a divisor of p^k whenever $\rho \neq 0$. Hence $|Y|$ is congruent to 1 mod p. Since $|Y|$ is a power of p, the subspace Y must be trivial. $\square$

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