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Artikel: ON THE CLASSIFICATION OF CERTAIN PIECEWISE LINEAR AND DIFFERENTIABLE MANIFOLDS IN DIMENSION EIGHT AND AUTOMORPHISMS OF $\sharp_{i=1}^b (S^2 \times S^5)$

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Kapitel: 5. Structure of the group $Aut_0^{PL}(\sharp_{i=1}^b (S^2 \times S^5))/Aut_0^{PL}(\sharp_{i=1}^b (S^2 \times D^6))$

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In other words, a manifold X with $\delta_X \equiv 0$ is piecewise linearly (smoothly) isomorphic $X^\dagger \# X^*$ where X^* is the type of X and $b_4(X^\dagger) = 0$. As our surgery arguments above reveal, an isomorphism between $X^\dagger \# X^*$ and $X'^\dagger \# X'^*$ can be chosen of the form $\varphi^\dagger \# \varphi^*$ where $\varphi^\dagger : X^\dagger \longrightarrow X'^\dagger$ and $\varphi^* : X^* \longrightarrow X'^*$ are isomorphisms. Therefore, the set of isomorphy classes of based piecewise linear E-manifolds of type X^* with $b_2 = b$ is in bijection to the set of isomorphy classes of based piecewise linear E-manifolds with $b_2 = b$ and $b_4 = 0$. The same goes for differentiable manifolds of type X^*, if X^* is not diffeomorphic to $X^* \# \Sigma$, Σ an exotic 8-sphere. Otherwise, we have to divide by the action of ϑ^8. This observation together with Corollary 4.9 settles Theorem 2.4. $\square$

5. STRUCTURE OF THE GROUP $\mathrm{Aut}_0^{\mathrm{PL}}\big(\#_{i=1}^b(S^2 \times S^5)\big) / \mathrm{Aut}_0^{\mathrm{PL}}\big(\#_{i=1}^b(S^2 \times D^6)\big)$

In this section we prove that $\mathrm{Aut}_0^{\mathrm{PL}}\big(\#_{i=1}^b(S^2 \times S^5)\big) / \mathrm{Aut}_0^{\mathrm{PL}}\big(\#_{i=1}^b(S^2 \times D^6)\big)$ is an abelian group which is, moreover, isomorphic to the group FL_b defined before. This result should be of some independent interest, especially because the group FL_b is quite well understood by Haefliger's work. For $b = 1$, we refer to [20] for more specific information.

We begin with the elementary

LEMMA 5.1. *Let $k \in \mathrm{Aut}_0^{\mathrm{PL}}\big(\#_{i=1}^b(S^2 \times S^5)\big)$ be a commutator. Then k extends to an automorphism of $\#_{i=1}^b(S^2 \times D^6)$.*

Proof. For the proof, we depict $\#_{i=1}^b(S^2 \times S^5)$ as follows: Let V_i, $i = 1, \dots, b$, be b copies of $S^2 \times D^6$, and we join V_i and V_{i+1} by a tube $T_i \cong [-1,1] \times D^7$, $i = 1, \dots, b-1$. The result is a manifold W whose boundary is isomorphic to $\#_{i=1}^b(S^2 \times S^5)$. We make the following normalizations: Write ∂V_i as $(S^2 \times D^i_+) \cup (S^2 \times D^i_-)$, let n_i and s_i be the centers of D^i_+ and D^i_-, respectively, and set $S^i_+ := S^2 \times n_i$ and $S^i_- := S^2 \times s_i$, $i = 1, \dots, b$. Choose furthermore points $e_i \neq w_i$ in $(S^2 \times D^i_+) \cap (S^2 \times D^i_-)$, $i = 1, \dots, b$, and suppose that $\{-1\} \times D^7 \subset T_i$ is attached to a disc around w_i in ∂V_i and $\{1\} \times D^7 \subset T_i$ to a disc around e_{i+1} in ∂V_{i+1}, $i = 1, \dots, b-1$. Set $T := \bigsqcup_{i=1}^{b-1} T_i$.

Now, let $k = f \circ g \circ f^{-1} \circ g^{-1}$ with $f, g \in \mathrm{Aut}_0^{\mathrm{PL}}\big(\#_{i=1}^b(S^2 \times S^5)\big)$. As $H_2(h, \mathbf{Z})$ is the identity for every element $h \in \mathrm{Aut}_0^{\mathrm{PL}}\big(\#_{i=1}^b(S^2 \times S^5)\big)$ and $S^i_\pm$, $i = 1, \dots, b$, both represent the same basis for $H_2(\partial W, \mathbf{Z})$, h is isotopic to a map h' which satisfies either assumption (A) or (B) below.

(A) : h' is trivial on a tubular neighborhood of S^i_+ which contains $(S^2 \times D^i_+) \setminus \mathrm{Int}(T)$, $i = 1, \ldots, b$.

(B) : h' is trivial on a tubular neighborhood of S^i_- which contains $(S^2 \times D^i_-) \setminus \mathrm{Int}(T)$, $i = 1, \ldots, b$.

Next, replace f by an isotopic map f' satisfying (A), and g by an isotopic map g' satisfying (B). Then k' is isotopic to $f' \circ g' \circ f'^{-1} \circ g'^{-1}$. The map k' is the identity outside $\mathrm{Int}(\partial T)$. It is, furthermore, the identity on a collar of $(\{-1\} \sqcup \{1\}) \times S^6$ in $R_i := [-1, 1] \times S^6 \subset \partial T_i$, $i = 1, \ldots, b-1$. Let k'_i be the restriction of k' to R_i, $i = 1, \ldots, b$. We know that each k'_i is the identity on a collar of $\{-1, 1\} \times S^6$ in R_i. Thus, we extend every k'_i to a PL automorphism $\widetilde{k}_i$ of $D^7 \times \{-1\} \cup R_i \cup D^7 \times \{1\} \cong S^7$ through $\mathrm{id}_{D^7 \times \{-1\} \cup D^7 \times \{1\}}$. Now, by [27], Lemma 1.10, p. 8, $\widetilde{k}_i$ extends to an automorphism κ_i of $D^8 \cong D^7 \times [-1, 1]$, $i = 1, \ldots, b$. Thus, the maps id_{V_i} and α_i, $i = 1, \ldots, b$, glue to an automorphism of $\#_{i=1}^b (S^2 \times D^6)$ whose restriction to the boundary is just k'. □

This lemma shows that $\mathrm{Aut}_0^{\mathrm{PL}}\big(\#_{i=1}^b (S^2 \times D^6)\big)$ is a normal subgroup of $\mathrm{Aut}_0^{\mathrm{PL}}\big(\#_{i=1}^b (S^2 \times S^5)\big)$, and that $\mathrm{Aut}_0^{\mathrm{PL}}\big(\#_{i=1}^b (S^2 \times S^5)\big) / \mathrm{Aut}_0^{\mathrm{PL}}\big(\#_{i=1}^b (S^2 \times D^6)\big)$ is abelian. Moreover, in Section 4.3, we have already defined a set theoretic bijection

$$\beta \colon \mathrm{Aut}_0^{\mathrm{PL}}\big(\#_{i=1}^b (S^2 \times S^5)\big) / \mathrm{Aut}_0^{\mathrm{PL}}\big(\#_{i=1}^b (S^2 \times D^6)\big) \longrightarrow \mathrm{FL}_b\,.$$

THEOREM 5.2. *The map β is a group isomorphism.*

Proof. Since β is bijective, we have to verify that β is a homomorphism. In order to do so, we will construct a group $\mathbf{G}$ together with surjective homomorphisms

$$\chi_1 \colon \mathbf{G} \longrightarrow \mathrm{Aut}_0^{\mathrm{PL}}\big(\#_{i=1}^b (S^2 \times S^5)\big) / \mathrm{Aut}_0^{\mathrm{PL}}\big(\#_{i=1}^b (S^2 \times D^6)\big)$$

and

$$\chi_2 \colon \mathbf{G} \longrightarrow \mathrm{FL}_b\,,$$

such that $\chi_2 = \beta \circ \chi_1$. This will clearly settle the claim.

Before we define $\mathbf{G}$, we recall some constructions and conventions from [11]. Let $S^8 = \{ (x_0, \ldots, x_9) \in \mathbf{R}^9 \mid x_0^2 + \cdots + x_9^2 = 1 \}$ be the unit sphere, write $S^8 = D^8_+ \cup D^8_-$, and let $\sigma \colon S^8 \longrightarrow S^8$ be the reflection at $S^7 = D^8_+ \cap D^8_-$, interchanging the Northern and the Southern hemispheres. First, let $S_b := (S^5_1, \ldots, S^5_b)$ be a 'standard link' in S^8 defined as follows: Fix real numbers $-1/2 < a_1 < \cdots < a_b < 1/2$, and set

$$S_i^5 := \left\{ (x_0, \ldots, x_9) \in S^8 \mid x_6 = x_7 = x_8 = 0,\ x_9 = a_i \right\}.$$

We choose, furthermore, framings $\tau_i \colon S_i^5 \times D^3 \longrightarrow S^8$ which extend over D^6, such that $\tau_i(D^5_{i,\pm} \times D^3) \subset D^8_\pm$ and $\sigma \circ \tau_i = \tau_i \circ (\sigma|_{S_i^5} \times \mathrm{id}_{D^3})$, $i = 1, \ldots, b$. Let l_b^0 be the resulting framed link in S^8 with $l^0_{b,\pm} := l_b^0 \cap D^8_\pm$. Recall from Section 1 of [11] that

1. Every framed link l of b five-dimensional spheres in S^8 is isotopic to a link l', such that either (A) $l' \cap D^8_+ = l^0_{b,+}$ or (B) $l' \cap D^8_- = l^0_{b,-}$.
2. If l_1 satisfies (A) and l_2 satisfies (B), then $l_1 + l_2$ is represented by the link l with $l \cap D^8_+ = l_2 \cap D^8_+$ and $l \cap D^8_- = l_1 \cap D^8_-$.

Note that, if we perform surgery along l_b^0, we get a manifold $W = W_+ \cup W_-$ which is isomorphic to $\#_{i=1}^b (S^2 \times S^6)$, and

$$W_\pm := \left(D^8_\pm \setminus \mathrm{Int}\,(l_b^0)\right) \cup \left(\bigsqcup_{i=1}^b (S_i^2 \times D^6_\pm)\right)$$

is canonically isomorphic to $\#_{i=1}^b (S^2 \times D^6)$. For the rest of the proof, we will use the description of $\#_{i=1}^b (S^2 \times S^5)$ as $\partial W_+ = \partial W_-$. Set

$$\mathbf{G} := \left\{ \text{PL-maps } f \colon S^7 \setminus \mathrm{Int}\,(l_b^0) \longrightarrow S^7 \setminus \mathrm{Int}\,(l_b^0) \colon f|_{\text{boundary}} = \mathrm{id} \right\}.$$

For every $f \in \mathbf{G}$, we define $\varphi(f) \colon \#_{i=1}^b (S^2 \times S^5) \longrightarrow \#_{i=1}^b (S^2 \times S^5)$, by extending f through the identity on $\bigsqcup_{i=1}^b (S_i^2 \times D^5)$. Similarly, define $\psi(f) \colon S^7 \longrightarrow S^7$. Obviously,

$$\begin{aligned} \chi_1 \colon \mathbf{G} &\longrightarrow \mathrm{Aut}_0^{\mathrm{PL}}\left(\#_{i=1}^b (S^2 \times S^5)\right) / \mathrm{Aut}_0^{\mathrm{PL}}\left(\#_{i=1}^b (S^2 \times D^6)\right) \\ f &\longmapsto [\varphi(f)] \end{aligned}$$

is a surjective homomorphism.

Next, we associate to $f \in \mathbf{G}$ an element $\chi_2(f) \in \mathrm{FL}_b$ as follows: First, we define $\Sigma(f) := D^8_+ \cup_{\psi(f)} D^8_-$ and the link $l'(f) := l^0_{b,+} \cup_{\psi(f)} l^0_{b,-}$. Then we choose a piecewise linear homeomorphism $F \colon \Sigma(f) \longrightarrow S^8$ and set $l_F(f) := F(l'(f))$. We have checked before that the isotopy class of $l_F(f)$ does not depend on the chosen homeomorphism, so that $\chi_2(f) := [l_F(f)] \in \mathrm{FL}_b$ is well defined. To see that $\chi_2 \colon \mathbf{G} \longrightarrow \mathrm{FL}_b$ is a homomorphism, let f, f' be in $\mathbf{G}$. Choose extensions $\overline{\psi} \colon D^8_+ \longrightarrow D^8_+$ and $\overline{\psi}' \colon D^8_- \longrightarrow D^8_-$ of $\psi(f)$ and $\psi(f')$, respectively. We then define $F \colon \Sigma(f) \longrightarrow S^8$ as $\overline{\psi}$ on D^8_+ and as the identity on D^8_-, $F' \colon \Sigma(f) \longrightarrow S^8$ as the identity on D^8_+ and $(\overline{\psi}')^{-1}$ on D^8_-, and $F'' \colon \Sigma(f' \circ f) \longrightarrow S^8$ as $\overline{\psi}$ on D^8_+ and $(\overline{\psi}')^{-1}$ on D^8_-. Then the link $l_F(f)$ satisfies (B), the link $l_{F'}(f')$ satisfies (A), and (2) above shows that $[l_{F''}(f' \circ f)] = [l_{F'}(f')] + [l_F(f)]$.

Finally, for given $f \in \mathbf{G}$, we can perform surgery on $\Sigma(f)$ along $l'(f)$. The result is $W_+ \cup_{\varphi(f)} W_-$. Reading this backwards means nothing else but $\beta(\chi_1(f)) = \chi_2(f)$ and we are done. $\square$