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DIFFERENTIABLE MANIFOLDS IN DIMENSION EIGHT AND AUTOMORPHISMS OF \$\sharp\_{i=1}^b (S^2 \times S^5)\$

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**Kapitel:** 5. Structure of the group \$Aut\_0^{PL}(\sharp\_{i=1}^b (S^2 \times

S^5))/Aut\_0^{PL}(\sharp\_{i=1}^b (S^2 \times D^6))\$

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In other words, a manifold X with  $\delta_X \equiv 0$  is piecewise linearly (smoothly) isomorphic  $X^\dagger \# X^*$  where  $X^*$  is the type of X and  $b_4(X^\dagger) = 0$ . As our surgery arguments above reveal, an isomorphism between  $X^\dagger \# X^*$  and  $X'^\dagger \# X'^*$  can be chosen of the form  $\varphi^\dagger \# \varphi^*$  where  $\varphi^\dagger \colon X^\dagger \longrightarrow X'^\dagger$  and  $\varphi^* \colon X^* \longrightarrow X'^*$  are isomorphisms. Therefore, the set of isomorphy classes of based piecewise linear E-manifolds of type  $X^*$  with  $b_2 = b$  is in bijection to the set of isomorphy classes of based piecewise linear E-manifolds with  $b_2 = b$  and  $b_4 = 0$ . The same goes for differentiable manifolds of type  $X^*$ , if  $X^*$  is not diffeomorphic to  $X^* \# \Sigma$ ,  $\Sigma$  an exotic 8-sphere. Otherwise, we have to divide by the action of  $\vartheta^8$ . This observation together with Corollary 4.9 settles Theorem 2.4.

## 5. Structure of the group $\operatorname{Aut}_0^{\operatorname{PL}}(\#_{i=1}^b(S^2\times S^5))/\operatorname{Aut}_0^{\operatorname{PL}}(\#_{i=1}^b(S^2\times D^6))$

In this section we prove that  $\operatorname{Aut}_0^{\operatorname{PL}}\left(\#_{i=1}^b(S^2\times S^5)\right)/\operatorname{Aut}_0^{\operatorname{PL}}\left(\#_{i=1}^b(S^2\times D^6)\right)$  is an abelian group which is, moreover, isomorphic to the group  $\operatorname{FL}_b$  defined before. This result should be of some independent interest, especially because the group  $\operatorname{FL}_b$  is quite well understood by Haefliger's work. For b=1, we refer to [20] for more specific information.

We begin with the elementary

LEMMA 5.1. Let  $k \in \operatorname{Aut}_0^{\operatorname{PL}}(\#_{i=1}^b(S^2 \times S^5))$  be a commutator. Then k extends to an automorphism of  $\#_{i=1}^b(S^2 \times D^6)$ .

*Proof.* For the proof, we depict  $\#_{i=1}^b(S^2\times S^5)$  as follows: Let  $V_i$ ,  $i=1,\ldots,b$ , be b copies of  $S^2\times D^6$ , and we join  $V_i$  and  $V_{i+1}$  by a tube  $T_i\cong [-1,1]\times D^7$ ,  $i=1,\ldots,b-1$ . The result is a manifold W whose boundary is isomorphic to  $\#_{i=1}^b(S^2\times S^5)$ . We make the following normalizations: Write  $\partial V_i$  as  $(S^2\times D_+^i)\cup (S^2\times D_-^i)$ , let  $n_i$  and  $s_i$  be the centers of  $D_+^i$  and  $D_-^i$ , respectively, and set  $S_+^i:=S^2\times n_i$  and  $S_-^i:=S^2\times s_i$ ,  $i=1,\ldots,b$ . Choose furthermore points  $e_i\neq w_i$  in  $(S^2\times D_+^i)\cap (S^2\times D_-^i)$ ,  $i=1,\ldots,b$ , and suppose that  $\{-1\}\times D^7\subset T_i$  is attached to a disc around  $w_i$  in  $\partial V_i$  and  $\{1\}\times D^7\subset T_i$  to a disc around  $e_{i+1}$  in  $\partial V_{i+1}$ ,  $i=1,\ldots,b-1$ . Set  $T:=\bigcup_{i=1}^{b-1}T_i$ .

Now, let  $k = f \circ g \circ f^{-1} \circ g^{-1}$  with  $f, g \in \operatorname{Aut}_0^{\operatorname{PL}}(\#_{i=1}^b(S^2 \times S^5))$ . As  $H_2(h, \mathbf{Z})$  is the identity for every element  $h \in \operatorname{Aut}_0^{\operatorname{PL}}(\#_{i=1}^b(S^2 \times S^5))$  and  $S^i_{\pm}$ ,  $i = 1, \ldots, b$ , both represent the same basis for  $H_2(\partial W, \mathbf{Z})$ , h is isotopic to a map h' which satisfies either assumption (A) or (B) below.

- (A): h' is trivial on a tubular neighborhood of  $S^i_+$  which contains  $(S^2 \times D^i_+) \setminus \operatorname{Int}(T), i = 1, \dots, b$ .
- (B): h' is trivial on a tubular neighborhood of  $S_{-}^{i}$  which contains  $(S^{2} \times D_{-}^{i}) \setminus \text{Int}(T), i = 1, ..., b$ .

Next, replace f by an isotopic map f' satisfying (A), and g by an isotopic map g' satisfying (B). Then k' is isotopic to  $f' \circ g' \circ f'^{-1} \circ g'^{-1}$ . The map k' is the identity outside  $\operatorname{Int}(\partial T)$ . It is, furthermore, the identity on a collar of  $(\{-1\} \sqcup \{1\}) \times S^6$  in  $R_i := [-1,1] \times S^6 \subset \partial T_i$ ,  $i=1,\ldots,b-1$ . Let  $k'_i$  be the restriction of k' to  $R_i$ ,  $i=1,\ldots,b$ . We know that each  $k'_i$  is the identity on a collar of  $\{-1,1\} \times S^6$  in  $R_i$ . Thus, we extend every  $k'_i$  to a PL automorphism  $\widetilde{k}_i$  of  $D^7 \times \{-1\} \cup R_i \cup D^7 \times \{1\} \cong S^7$  through  $\operatorname{id}_{D^7 \times \{-1\} \cup D^7 \times \{1\}}$ . Now, by [27], Lemma 1.10, p. 8,  $\widetilde{k}_i$  extends to an automorphism  $\kappa_i$  of  $D^8 \cong D^7 \times [-1,1]$ ,  $i=1,\ldots,b$ . Thus, the maps  $\operatorname{id}_{V_i}$  and  $\alpha_i$ ,  $i=1,\ldots,b$ , glue to an automorphism of  $\#_{i=1}^b(S^2 \times D^6)$  whose restriction to the boundary is just k'.  $\square$ 

This lemma shows that  $\operatorname{Aut}_0^{\operatorname{PL}}\left(\#_{i=1}^b(S^2\times D^6)\right)$  is a normal subgroup of  $\operatorname{Aut}_0^{\operatorname{PL}}\left(\#_{i=1}^b(S^2\times S^5)\right)$ , and that  $\operatorname{Aut}_0^{\operatorname{PL}}\left(\#_{i=1}^b(S^2\times S^5)\right)/\operatorname{Aut}_0^{\operatorname{PL}}\left(\#_{i=1}^b(S^2\times D^6)\right)$  is abelian. Moreover, in Section 4.3, we have already defined a set theoretic bijection

$$\beta: \operatorname{Aut}_0^{\operatorname{PL}}(\#_{i=1}^b(S^2 \times S^5)) / \operatorname{Aut}_0^{\operatorname{PL}}(\#_{i=1}^b(S^2 \times D^6)) \longrightarrow \operatorname{FL}_b$$
.

THEOREM 5.2. The map  $\beta$  is a group isomorphism.

*Proof.* Since  $\beta$  is bijective, we have to verify that  $\beta$  is a homomorphism. In order to do so, we will construct a group G together with surjective homomorphisms

$$\chi_1: \mathbf{G} \longrightarrow \operatorname{Aut}_0^{\operatorname{PL}} \left( \#_{i=1}^b (S^2 \times S^5) \right) / \operatorname{Aut}_0^{\operatorname{PL}} \left( \#_{i=1}^b (S^2 \times D^6) \right)$$

and

$$\chi_2 \colon \mathbf{G} \longrightarrow \mathrm{FL}_b$$
,

such that  $\chi_2 = \beta \circ \chi_1$ . This will clearly settle the claim.

Before we define G, we recall some constructions and conventions from [11]. Let  $S^8 = \{(x_0, \ldots, x_9) \in \mathbb{R}^9 \mid x_0^2 + \cdots + x_9^2 = 1\}$  be the unit sphere, write  $S^8 = D_+^8 \cup D_-^8$ , and let  $\sigma \colon S^8 \longrightarrow S^8$  be the reflection at  $S^7 = D_+^8 \cap D_-^8$ , interchanging the Northern and the Southern hemispheres. First, let  $S_b := (S_1^5, \ldots, S_b^5)$  be a 'standard link' in  $S^8$  defined as follows: Fix real numbers  $-1/2 < a_1 < \cdots < a_b < 1/2$ , and set

$$S_i^5 := \{ (x_0, \dots, x_9) \in S^8 \mid x_6 = x_7 = x_8 = 0, \ x_9 = a_i \}.$$

We choose, furthermore, framings  $\tau_i \colon S_i^5 \times D^3 \longrightarrow S^8$  which extend over  $D^6$ , such that  $\tau_i(D_{i,\pm}^5 \times D^3) \subset D_{\pm}^8$  and  $\sigma \circ \tau_i = \tau_i \circ (\sigma|_{S_i^5} \times \mathrm{id}_{D^3}), \ i = 1, \ldots, b$ . Let  $l_b^0$  be the resulting framed link in  $S^8$  with  $l_{b,\pm}^0 := l_b^0 \cap D_{\pm}^8$ . Recall from Section 1 of [11] that

- 1. Every framed link l of b five-dimensional spheres in  $S^8$  is isotopic to a link l', such that either (A)  $l' \cap D^8_+ = l^0_{b,+}$  or (B)  $l' \cap D^8_- = l^0_{b,-}$ .
- 2. If  $l_1$  satisfies (A) and  $l_2$  satisfies (B), then  $l_1 + l_2$  is represented by the link l with  $l \cap D_+^8 = l_2 \cap D_+^8$  and  $l \cap D_-^8 = l_1 \cap D_-^8$ .

Note that, if we perform surgery along  $l_b^0$ , we get a manifold  $W = W_+ \cup W_-$  which is isomorphic to  $\#_{i=1}^b(S^2 \times S^6)$ , and

$$W_{\pm} := \left(D_{\pm}^8 \setminus \operatorname{Int}(l_b^0)\right) \cup \left(\bigsqcup_{i=1}^b (S_i^2 \times D_{\pm}^6)\right)$$

is canonically isomorphic to  $\#_{i=1}^b(S^2 \times D^6)$ . For the rest of the proof, we will use the description of  $\#_{i=1}^b(S^2 \times S^5)$  as  $\partial W_+ = \partial W_-$ . Set

$$\mathbf{G} := \{ \operatorname{PL-maps} f : S^7 \setminus \operatorname{Int}(l_b^0) \longrightarrow S^7 \setminus \operatorname{Int}(l_b^0) : f|_{\text{boundary}} = \operatorname{id} \} .$$

For every  $f \in \mathbf{G}$ , we define  $\varphi(f) \colon \#_{i=1}^b(S^2 \times S^5) \longrightarrow \#_{i=1}^b(S^2 \times S^5)$ , by extending f through the identity on  $\bigsqcup_{i=1}^b(S_i^2 \times D^5)$ . Similarly, define  $\psi(f) \colon S^7 \longrightarrow S^7$ . Obviously,

$$\chi_1 : \mathbf{G} \longrightarrow \operatorname{Aut}_0^{\operatorname{PL}} \left( \#_{i=1}^b (S^2 \times S^5) \right) / \operatorname{Aut}_0^{\operatorname{PL}} \left( \#_{i=1}^b (S^2 \times D^6) \right)$$

$$f \longmapsto [\varphi(f)]$$

is a surjective homomorphism.

Next, we associate to  $f \in \mathbf{G}$  an element  $\chi_2(f) \in \mathrm{FL}_b$  as follows: First, we define  $\Sigma(f) := D_+^8 \cup_{\psi(f)} D_-^8$  and the link  $l'(f) := l_{b,+}^0 \cup_{\psi(f)} l_{b,-}^0$ . Then we choose a piecewise linear homeomorphism  $F \colon \Sigma(f) \longrightarrow S^8$  and set  $l_F(f) := F(l'(f))$ . We have checked before that the isotopy class of  $l_F(f)$  does not depend on the chosen homeomorphism, so that  $\chi_2(f) := [l_F(f)] \in \mathrm{FL}_b$  is well defined. To see that  $\chi_2 \colon \mathbf{G} \longrightarrow \mathrm{FL}_b$  is a homomorphism, let f, f' be in  $\mathbf{G}$ . Choose extensions  $\overline{\psi} \colon D_+^8 \longrightarrow D_+^8$  and  $\overline{\psi}' \colon D_-^8 \longrightarrow D_-^8$  of  $\psi(f)$  and  $\psi(f')$ , respectively. We then define  $F \colon \Sigma(f) \longrightarrow S^8$  as  $\overline{\psi}$  on  $D_+^8$  and as the identity on  $D_-^8$ ,  $F' \colon \Sigma(f) \longrightarrow S^8$  as  $\overline{\psi}$  on  $D_+^8$  and  $(\overline{\psi}')^{-1}$  on  $D_-^8$ , and  $F'' \colon \Sigma(f' \circ f) \longrightarrow S^8$  as  $\overline{\psi}$  on  $D_+^8$  and  $(\overline{\psi}')^{-1}$  on  $D_-^8$ . Then the link  $l_F(f)$  satisfies (B), the link  $l_{F'}(f')$  satisfies (A), and (2) above shows that  $[l_{F''}(f' \circ f)] = [l_{F'}(f')] + [l_F(f)]$ .

Finally, for given  $f \in \mathbf{G}$ , we can perform surgery on  $\Sigma(f)$  along l'(f). The result is  $W_+ \cup_{\varphi(f)} W_-$ . Reading this backwards means nothing else but  $\beta(\chi_1(f)) = \chi_2(f)$  and we are done.  $\square$