4. Proof of Theorem 2.2 and Theorem 2.4

From now on, X stands for an eight-dimensional E-manifold with $w_2(X) = 0$.

4.1 Proof of Theorem 2.2

The classification result for 3-connected E-manifolds of dimension eight is a special case of a result of Wall's [36] and can be easily obtained with the methods described in [17], VII, § 12. Let us recall the details, because we will need them later on.

We fix a basis $\underline{b}$ for $H_4(X, \mathbf{Z})$ and let $\underline{y}$ be the dual basis of $H^4(X, Z)$. Then there is a handle presentation $X = \overline{D}^8 \cup H_1^4 \cup \cdots \cup H_{b'}^4 \cup D^8$ with $\underline{b}$ as the preferred basis. The manifold $T := D^8 \cup H_1^4 \cup \cdots \cup H_{b'}^4$ is determined by the ambient isotopy class of a framed link of 3-spheres in S^7, having b' components. Let us first look at such a link, forgetting the framing, i.e., suppose we are given embeddings $g_i \colon S^3 \longrightarrow S^7$ with $S_i \cap S_j = \varnothing$ for $i \neq j$, $S_i := g_i(S^3)$, $i = 1, \ldots, b'$. By 3.9, we may assume that the g_i are differentiable. Observe that the normal bundles of the S_i are trivial.

We equip S_i with the orientation induced via g_i by the standard orientation of S^3 and the normal bundle of S_i with the orientation which is determined by requiring that the orientation of S_i followed by that of its normal bundle coincide with the orientation of S^7. Therefore, a 3-sphere F_i which bounds the fibre of a tubular neighborhood of S_i in S^7 inherits an orientation and thus provides a generator e_i for $H_3(S^7 \setminus S_i, \mathbf{Z}) \cong \mathbf{Z}$, $i = 1, \ldots, b'$. For $i \neq j$, the image of the fundamental class $[S_i]$ in $H_3(S^7 \setminus S_j, \mathbf{Z})$ is of the form $\lambda_{ij} \cdot e_j$. The integer λ_{ij} is called *the linking number of S_i and S_j*.

For $i = 1, \ldots, b'$, the manifold $S^7 \setminus \bigcup_{j \neq i} S_j$ is up to dimension 5 homotopy equivalent to $\bigvee_{j \neq i} F_j$, and

$$\pi_3\big(S^7 \setminus \bigcup_{j \neq i} S_j\big) \cong \pi_3\big(\bigvee_{j \neq i} F_j\big) \cong \bigoplus_{j \neq i} H_3\big(S^7 \setminus S_j, \mathbf{Z}\big) .$$

Under this identification, we have $[g_i] = \sum_{j \neq i} \lambda_{ij} \cdot e_j$. The $[g_i]$ determine the ambient isotopy class of the given link (3.9), and we deduce

Proposition 4.1. *The linking numbers λ_{ij}, $1 \leq i < j \leq b'$, determine the given link up to ambient isotopy.*

The sphere S_i bounds a 4-dimensional disc D_i^- in D^8, $i = 1, \ldots, b'$, which we equip with the induced orientation. We may, furthermore, assume that the D_i^- intersect transversely in the interior of D^8. Then the λ_{ij} coincide with the intersection numbers $D_i^-.D_j^-$, $1 \leq i < j \leq b'$. For an intuitive proof (in dimension 4), see [28], p. 67. Now, every disc D_i^- is completed by the core disc D_i^+ of the i^{th} 4-handle to an embedded 4-sphere Σ_i in T, $i = 1, \ldots, b'$, and, since all the core discs are pairwise disjoint, the λ_{ij} coincide with the intersection numbers $\Sigma_i.\Sigma_j$, $1 \leq i < j \leq b'$. Finally, X is obtained by gluing an 8-disc to T along ∂T, and the spheres Σ_i represent the elements of the chosen basis $\underline{b}$ of $H_4(X, \mathbf{Z})$. Identifying the intersection ring with the cohomology ring of X via Poincaré-duality, we see

COROLLARY 4.2. *The linking numbers λ_{ij} coincide with the cup products $(y_i \cup y_j)[X]$, $1 \leq i < j \leq b'$, i.e., the link of the attaching spheres is determined up to ambient isotopy by the basis $\underline{b}$ and the cup products.*

As we have remarked before, the normal bundles of the S_i are trivial, whence there exist embeddings $f_i^0 \colon S^3 \times D^4 \longrightarrow S^7$ with $f^0|_{S^3 \times \{0\}} = g_i$, $i = 1, \ldots, b'$. From the uniqueness of tubular (in differential topology) or regular (in piecewise linear topology) neighbourhoods, every other embedding $f_i \colon S^3 \times D^4 \longrightarrow S^7$ with $f_i|_{S^3 \times \{0\}} = g_i$ is ambient isotopic to one of the form $f_i^{[h_i]} := \big((x, y) \longmapsto (x, h_i \cdot y)\big)$, $[h_i] \in \pi_3(\mathrm{SO}(4))$, $i = 1, \ldots, b'$. Corollary 3.14 implies that we can choose the f_i^0, $i = 1, \ldots, b'$, in such a way that the following holds:

LEMMA 4.3. *Suppose T is obtained by attaching 4-handles along $f_i^{[h_i]}$ with $[h_i] = k_1^i \alpha_3 + k_2^i \beta_3$, $i = 1, \ldots, b'$, then*

$$\Sigma_i.\Sigma_i = k_2^i \quad \textit{and} \quad p_1(T_{T|\Sigma_i}) = \pm\big(2k_2^i + 4k_1^i\big)\,.$$

This shows that also the framed link used for constructing T and X is determined by the system of invariants associated to $(X, \underline{y})$, proving the injectivity in Part i) of the theorem. Moreover, the assertion about the fibres in Part ii) is clear.

Conversely, given a system Z of invariants in $Z(0, b')$, satisfying relation (2), there exists a based 3-connected manifold $(X, \underline{y})$ realizing Z. Indeed, by the above identification of the invariants, Z determines a framed link in S^7 and thus the manifold $T := D^8 \cup H_1^4 \cup \cdots \cup H_{b'}^4$. The boundary of T is a 7-dimensional homotopy sphere ([17], (12.2), p. 119) and, therefore, piecewise linearly homeomorphic to S^7. Hence, $X = T \cup_{S^7} D^8$ is a piecewise linear manifold with the desired system of invariants, settling Part i). If, in addition, relation (3) holds, then [18] ensures that X will carry a smooth structure (compare Theorem A.4 of [24]), finishing the proof of Part ii). □

4.2 THE DETERMINATION OF W_4 IN THE GENERAL CASE

We have a handle decomposition $W_0 \subset W_2 \subset W_4 \subset W_6 \subset X$ of X providing preferred bases $\underline{b}$ of $H_2(X,\mathbf{Z})$ and $\underline{c}$ of $H_4(X,\mathbf{Z})$, respectively. Let $\underline{x}$ and $\underline{y}$ be the dual bases of $H^2(X,\mathbf{Z})$ and $H^4(X,\mathbf{Z})$, respectively. Finally, let $\underline{y}^*$ be the basis of $H^4(X,\mathbf{Z})$ which is dual to $\underline{y}$ via γ_X.

We find $\partial W_2 \cong \#_{i=1}^{b}(S^2 \times S^5)$, and W_4 is determined by the ambient isotopy class of a framed link of 3-spheres in ∂W_2 with b' components. Let $f_k\colon S^3 \times D^4 \longrightarrow \partial W_2$ be the k^{th} component of that link and $g_k := f_k|_{S^3\times\{0\}}$, $k = 1,\dots,b'$. In the notation of Section 3.6, we write $[g_k] \in \pi_3(\partial W_2 \setminus \bigcup_{k\neq j} S_j)$ in the form $(l_i^k, i = 1,\dots,b, l_{ij}^k, 1 \le i < j \le b; \lambda_{kj}, j \neq k)$, $k = 1,\dots,b'$. To see the significance of the l_i^k and l_{ij}^k, note that, by Remark 3.4, $W_2 \cup H_k^4 \subset X$ is homotopy equivalent to $\left(\bigvee_{i=1}^{b} S^2\right) \cup_{g_k} D^4$. The cohomology ring of that complex has been computed in Proposition 3.11, so that the naturality of the cup product implies the following formulae for the cup products in X:

$$x_i \cup x_j = \sum_{k=1}^{b'} l_{ij}^k \cdot y_k^*, \quad i \neq j,$$

$$x_i \cup x_i = \sum_{k=1}^{b'} l_i^k \cdot y_k^*, \quad i = 1,\dots,b.$$

Therefore, the l_i^k and l_{ij}^k are determined by δ_X and γ_X (used to compute $\underline{y}^*$), in fact $l_i^k = \gamma_X(\delta(x_i \otimes x_i) \otimes y_k)$ and $l_{ij}^k = \gamma_X(\delta(x_i \otimes x_j) \otimes y_k)$.

To determine the λ_{ij} and the framings, we proceed as follows: Look at the embedding $\#_{i=1}^{b}(S^2 \times S^5) \hookrightarrow X$. There exist b embedded 2-spheres $S_1^2,\dots,S_b^2$ which represent the basis $\underline{b}$ and which do not meet the given link. Finally, $\#_{i=1}^{b}(S^2 \times S^5)$ obviously possesses a regular neighborhood in X which is homeomorphic to $\#_{i=1}^{b}(S^2 \times S^5) \times D^1$. Thus, we can perform "surgery in pairs" as described in Section 3.1. The result is a 3-connected manifold X^* containing S^7. It is by construction the manifold obtained from the framed link in S^7 derived from the given one in $\#_{i=1}^{b}(S^2 \times S^5)$ (cf. Section 4.1). We will be finished, once we are able to compare the invariants of X to those of X^*. To do so, we look at the *trace of the surgery*, i.e., at $Y = (X \times I) \cup H_1^5 \cup \dots \cup H_{b'}^5$, the 5-handles being attached along tubular neighborhoods of the $S_i \times \{1\}$ in $X \times \{1\}$. Then $\partial Y = X \sqcup \overline{X}^*$.

The Mayer-Vietoris sequence provides the isomorphisms

$$H_4(X,\mathbf{Z}) \cong H_4\bigl(X \setminus \bigsqcup_{i=1}^{b'} (S_i \times D^6), \mathbf{Z}\bigr) \cong H_4(X^*, \mathbf{Z}).$$

Set $H := H_4\bigl(X \setminus \bigsqcup_{i=1}^{b'} (S_i \times D^6), \mathbf{Z}\bigr)$. By Lefschetz duality ([5], (28.18)), there is for each $q \in \mathbf{N}$ a diagram (omitting $\mathbf{Z}$-coefficients)

$$\begin{tikzcd}
H^{q-1}(Y) \arrow[r] \arrow[d, "\cong"] & H^{q-1}(\partial Y) \arrow[r] \arrow[d, "\cong"] & H^q(Y, \partial Y) \arrow[r] \arrow[d, "\cong"] & H^q(Y) \arrow[d, "\cong"] \\
H_{10-q}(Y, \partial Y) \arrow[r] & H_{9-q}(\partial Y) \arrow[r] & H_{9-q}(Y) \arrow[r] & H_{9-q}(Y, \partial Y)
\end{tikzcd} \tag{4}$$

where the left square commutes up to the sign $(-1)^{q-1}$ and the other two commute. We first use it in the case $q = 5$. Look at the commutative diagram

$$\begin{tikzcd}
H \arrow[r, "\cong"] \arrow[d, "\cong"] & H_4(X^*, \mathbf{Z}) \arrow[d] \\
H_4(X, \mathbf{Z}) \arrow[r] & H_4(Y, \mathbf{Z}),
\end{tikzcd}$$

in which all arrows are injective, because $H_5(Y, X; \mathbf{Z}) = 0 = H_5(Y, X^*; \mathbf{Z})$ (cf. [17], p. 198). Using the identification $H_4(\partial Y, \mathbf{Z}) = H \oplus H$, we find

$$\operatorname{Im}\bigl(H_5(Y, \partial Y; \mathbf{Z})\bigr) = \{\, (y, -y) \in H \oplus H \,\}. \tag{5}$$

Similar considerations apply to the case $q = 9$. Taking into account that X^* sits in Y with the reversed orientation, (4) shows that the forms γ_X and γ_{X^*}, both defined with respect to the preferred bases, coincide. In the same manner, the pullbacks of $p_1(Y)$ to $H^4(X, \mathbf{Z})$ and $H^4(X^*, \mathbf{Z})$, respectively, agree. Since X and X^* are the boundary components of Y, these pullbacks are $p_1(X)$ and $p_1(X^*)$, respectively, and we are done. $\square$

4.3 MANIFOLDS WITH GIVEN INVARIANTS

One might speculate, especially in view of the classification of E-manifolds in dimension 4 and 6, that the invariants δ_X, γ_X, and $p_1(X)$ might suffice to classify E-manifolds with $w_2(X) = 0$ in dimension 8. However, Lemma 3.6 shows that these invariants determine only W_4 and we still have the choice of an isomorphism in gluing $\#_{i=1}^{b}(S^2 \times S^5)$ to W_4, and different gluings may lead to different results. The following example, which was communicated to me by J.-C. Hausmann, illustrates this phenomenon.

EXAMPLE 4.4. One has $\pi_5(\mathrm{SO}(3)) \cong \mathbf{Z}_2$ [32]. Therefore, there are two different S^2-bundles over S^6, call them $X := S^6 \times S^2$ and $X' := S^6 \tilde{\times} S^2$. Obviously, X and X' are spin-manifolds with trivial invariants, but one computes $\pi_5(X) \cong \mathbf{Z}_2$ and $\pi_5(X') = \{0\}$.

Fix b, b', and a system Z of invariants in the image of the map $Z^{\mathrm{PL}(C^\infty)}(b, b')$. As we have seen, Z determines a certain manifold W_4 whose boundary is diffeomorphic to $\#_{i=1}^{b}(S^2 \times S^5)$ together with a basis $\underline{b}$ for $H_2(\partial W_4, \mathbf{Z})$. Let $\underline{b}_0$ be the natural basis for $H_2(\#_{i=1}^{b}(S^2 \times S^5), \mathbf{Z})$, and denote by $\mathrm{Iso}_0^{\mathrm{PL}(C^\infty)}$ the set of piecewise linear (smooth) isomorphisms $f\colon \#_{i=1}^{b}(S^2 \times S^5) \longrightarrow \partial W_4$ with $f_*(\underline{b}_0) = \underline{b}$. Our results show that every based piecewise linear (smooth) manifold $(X, \underline{x}, \underline{y})$ with system of invariants Z is piecewise linearly (smoothly) isomorphic to a manifold of the form

$$X(f) := \partial W_4 \cup_f \#_{i=1}^{b}(S^2 \times S^5) \quad \text{for some } f \in \mathrm{Iso}_0^{\mathrm{PL}(C^\infty)}$$

with its given bases for $H^2(X(f), \mathbf{Z})$ and $H^4(X(f), \mathbf{Z})$. Conversely, every manifold of the form $X(f)$ is a piecewise linear (smooth) based E-manifold with invariants Z.

Now, suppose we are given $f, f' \in \mathrm{Iso}_0^{\mathrm{PL}(C^\infty)}$, such that $X(f)$ and $X(f')$ are isomorphic as piecewise linear (smooth) based manifolds. We claim that we can find an isomorphism $\varphi\colon X(f) \longrightarrow X(f')$ with $\varphi(W_4) = W_4$. For this, look at the handle decomposition $W_0 \subset W_2 \subset W_4$. Since W_0 is just an embedded 8-disc in $X(f)$ and $X(f')$, respectively, we can choose φ with $\varphi(W_0) = W_0$. Let $l \subset \partial W_0$ be the framed link for attaching the 2-handles. Then $\varphi(l)$ and l are isotopic. Therefore, we can find a level preserving diffeomorphism $\tilde{\psi}\colon [-1,1] \times \partial W_0 \longrightarrow [-1,1] \times \partial W_0$ with $\tilde{\psi}|_{\{\pm 1\} \times \partial W_0} = \mathrm{id}_{\partial W_0}$ and $\tilde{\psi}|_{\{0\} \times \partial W_0}(\varphi(l)) = l$. If we choose a tubular neighborhood ($\cong [-1,1] \times \partial W_0$) of ∂W_0 in $X(f')$, we can use $\tilde{\psi}$ to define an automorphism $\psi\colon X(f') \longrightarrow X(f')$ with $\psi(\varphi(l)) = l$. Thus, $\psi \circ \varphi$ maps W_2 onto W_2. A similar argument shows that we can achieve $\varphi(W_4) = W_4$.

Let $\mathrm{Aut}_0^{\mathrm{PL}(C^\infty)}(\#_{i=1}^b(S^2 \times D^6))$ be the group of piecewise linear (smooth) automorphisms g of $\#_{i=1}^b(S^2 \times D^6)$ with $H^2(g, \mathbf{Z}) = \mathrm{id}$ and similarly define $\mathrm{Aut}_0^{\mathrm{PL}(C^\infty)}(W_4)$. Then we have just established

PROPOSITION 4.5. *The set of isomorphy classes of based piecewise linear (smooth) E-manifolds with invariants Z is in bijection to the set of equivalence classes in* $\mathrm{Iso}_0^{\mathrm{PL}(C^\infty)}$ *with respect to the equivalence relation coming from the group action*

$$\mathrm{Aut}_0^{\mathrm{PL}(C^\infty)}(W_4) \times \mathrm{Aut}_0^{\mathrm{PL}(C^\infty)}(\#_{i=1}^b(S^2 \times D^6)) \times \mathrm{Iso}_0^{\mathrm{PL}(C^\infty)} \longrightarrow \mathrm{Iso}_0^{\mathrm{PL}(C^\infty)}$$
$$(h, g, f) \longmapsto h|_{\partial W_4} \circ f \circ g|^{-1}_{\#_{i=1}^b(S^2 \times S^5)} .$$

We shall see in Lemma 5.1 that $\mathrm{Aut}_0^{\mathrm{PL}}(\#_{i=1}^b(S^2 \times D^6))$ contains the commutator subgroup of $\mathrm{Aut}_0^{\mathrm{PL}}(\#_{i=1}^b(S^2 \times S^5))$.

COROLLARY 4.6. *The set of isomorphy classes of based piecewise linear E-manifolds with* $b_2 = b$ *and* $b_4 = 0$ *is in bijection to the abelian group*

$$\mathrm{Aut}_0^{\mathrm{PL}}(\#_{i=1}^b(S^2 \times S^5)) / \mathrm{Aut}_0^{\mathrm{PL}}(\#_{i=1}^b(S^2 \times D^6)) .$$

I have been informed by experts that the structure of the groups $\mathrm{Aut}_0^{\mathrm{PL}(C^\infty)}(\#_{i=1}^b(S^2 \times S^5))$ and $\mathrm{Aut}_0^{\mathrm{PL}(C^\infty)}(\#_{i=1}^b(S^2 \times D^6))$ has not yet been determined and that this would be a rather difficult task. Therefore, we choose the viewpoint of framed links in order to finish our considerations. In Theorem 5.2, we will then use this viewpoint to compute the group $\mathrm{Aut}_0^{\mathrm{PL}}(\#_{i=1}^b(S^2 \times S^5)) / \mathrm{Aut}_0^{\mathrm{PL}}(\#_{i=1}^b(S^2 \times D^6))$.

As above, let $(X, \underline{x}, \underline{y})$ be a based piecewise linear (smooth) E-manifold with zero second Stiefel-Whitney class and system of invariants $Z_{(X,\underline{x},\underline{y})} = (\delta, \gamma, p)$. We have seen that we can find a framed link l_X of 2-spheres in X which represents the basis $\underline{x}$ and perform surgery along this link in order to get a 3-connected piecewise linear (smooth) based manifold $(X^*, \underline{y})$ together with a framed link $l^*_{X'}$ of 5-spheres in it. If $(X', \underline{x}', \underline{y}', l_{X'})$ is another such object where $(X', \underline{x}', \underline{y}')$ is isomorphic to $(X, \underline{x}, \underline{y})$, then clearly we can find an isomorphism $\varphi\colon (X, \underline{x}, \underline{y}) \longrightarrow (X', \underline{x}', \underline{y}')$ with $\varphi(l_X) = l_{X'}$. Such an isomorphism φ yields, after surgery, an isomorphism $\varphi^*\colon (X^*, \underline{y}) \longrightarrow (X'^*, \underline{y}')$ with $\varphi^*(l^*_{X'}) = l^*_{X'^*}$. In particular, the manifold $(X^*, \underline{y})$ is determined up to piecewise linear (smooth) isomorphy. We call it the *type of* $(X, \underline{x}, \underline{y})$. Note that this notion matters only in the smooth case, by Theorem 2.2.

To summarize, we have

PROPOSITION 4.7. *The set of isomorphy classes of based piecewise linear (smooth) E-manifolds of type* $(X^*, \underline{y})$ *is in bijection to the set of equivalence classes of framed links of* 5*-spheres in* X^* *where two such links* l *and* l' *are considered* equivalent, *if there is a piecewise linear (smooth) automorphism* $\varphi^* \colon (X^*, \underline{y}) \longrightarrow (X^*, \underline{y})$ *with* $\varphi^*(l) = l'$.

EXAMPLE 4.8. The group $\mathbf{Z}_2^{\oplus b}$ acts freely on the set of isotopy classes of framed links of b spheres of dimension 5 in X^* by altering the framings of the components. Note that the two possible framings of the trivial bundle on a 5-sphere are distinguished by the fact that one extends over D^6 and the other does not. This property is preserved under piecewise linear homeomorphisms, so that we conclude that $\mathbf{Z}_2^{\oplus b}$ acts also freely on the set of equivalence classes of framed links of b spheres of dimension 5 in X^*.

Note that this completes the classification of Spin-E-manifolds of dimension eight with second Betti number one.

Let us look at manifolds of type S^8. We claim that two framed links l and l' of 5-spheres are equivalent in the above sense, if and only if they are isotopic. Clearly, after replacing l and l' by isotopic links, we may assume that both of them are contained in the Southern hemisphere and that φ^* is the identity on the Northern hemisphere. Now, choose a representative $\varphi^\dagger$ for the isotopy class of ${\varphi^*}^{-1}$ which is the identity on the Southern hemisphere. Then $\varphi^\dagger \circ \varphi^*$ is isotopic to the identity and carries l into l'.

For differentiable manifolds, the operation $X \longmapsto X\#\Sigma$, Σ an exotic 8-sphere, establishes a bijection between the set of isomorphy classes of based smooth E-manifolds of type S^8 and the set of isomorphy classes of based smooth E-manifolds of type Σ. We conclude

COROLLARY 4.9. i) *The set of isomorphy classes of based piecewise linear E-manifolds with* $b_2 = b$ *and* $b_4 = 0$ *is in bijection to the group* $\mathrm{FL}_b = \mathrm{L}_b \oplus \bigoplus_{i=1}^b \mathbf{Z}_2$.

ii) *The set of isomorphy classes of based smooth E-manifolds with* $b_2 = b$ *and* $b_4 = 0$ *is in bijection to the group* $\vartheta^8 \oplus \mathrm{FL}_b$.

Finally, we have to deal with those manifolds for which the cup form δ is trivial. Our investigations in Sections 3.6 and 4.2 show that the framed link of 3-spheres in ∂W_2 can be chosen to be contained in a small disc.

In other words, a manifold X with $\delta_X \equiv 0$ is piecewise linearly (smoothly) isomorphic $X^{\dagger}\#X^*$ where X^* is the type of X and $b_4(X^{\dagger}) = 0$. As our surgery arguments above reveal, an isomorphism between $X^{\dagger}\#X^*$ and $X'^{\dagger}\#X'^*$ can be chosen of the form $\varphi^{\dagger}\#\varphi^*$ where $\varphi^{\dagger}\colon X^{\dagger} \longrightarrow X'^{\dagger}$ and $\varphi^*\colon X^* \longrightarrow X'^*$ are isomorphisms. Therefore, the set of isomorphy classes of based piecewise linear E-manifolds of type X^* with $b_2 = b$ is in bijection to the set of isomorphy classes of based piecewise linear E-manifolds with $b_2 = b$ and $b_4 = 0$. The same goes for differentiable manifolds of type X^*, if X^* is not diffeomorphic to $X^*\#\Sigma$, Σ an exotic 8-sphere. Otherwise, we have to divide by the action of ϑ^8. This observation together with Corollary 4.9 settles Theorem 2.4. $\square$

5. STRUCTURE OF THE GROUP $\mathrm{Aut}_0^{\mathrm{PL}}\big(\#_{i=1}^b(S^2 \times S^5)\big)/\,\mathrm{Aut}_0^{\mathrm{PL}}\big(\#_{i=1}^b(S^2 \times D^6)\big)$

In this section we prove that $\mathrm{Aut}_0^{\mathrm{PL}}\big(\#_{i=1}^b(S^2 \times S^5)\big)/\,\mathrm{Aut}_0^{\mathrm{PL}}\big(\#_{i=1}^b(S^2 \times D^6)\big)$ is an abelian group which is, moreover, isomorphic to the group FL_b defined before. This result should be of some independent interest, especially because the group FL_b is quite well understood by Haefliger's work. For $b = 1$, we refer to [20] for more specific information.

We begin with the elementary

LEMMA 5.1. *Let $k \in \mathrm{Aut}_0^{\mathrm{PL}}\big(\#_{i=1}^b(S^2 \times S^5)\big)$ be a commutator. Then k extends to an automorphism of $\#_{i=1}^b(S^2 \times D^6)$.*

Proof. For the proof, we depict $\#_{i=1}^b(S^2 \times S^5)$ as follows: Let V_i, $i = 1,\dots,b$, be b copies of $S^2 \times D^6$, and we join V_i and V_{i+1} by a tube $T_i \cong [-1,1]\times D^7$, $i = 1,\dots,b-1$. The result is a manifold W whose boundary is isomorphic to $\#_{i=1}^b(S^2 \times S^5)$. We make the following normalizations: Write ∂V_i as $(S^2 \times D^i_+) \cup (S^2 \times D^i_-)$, let n_i and s_i be the centers of D^i_+ and D^i_-, respectively, and set $S^i_+ := S^2 \times n_i$ and $S^i_- := S^2 \times s_i$, $i = 1,\dots,b$. Choose furthermore points $e_i \neq w_i$ in $(S^2 \times D^i_+) \cap (S^2 \times D^i_-)$, $i = 1,\dots,b$, and suppose that $\{-1\} \times D^7 \subset T_i$ is attached to a disc around w_i in ∂V_i and $\{1\} \times D^7 \subset T_i$ to a disc around e_{i+1} in ∂V_{i+1}, $i = 1,\dots,b-1$. Set $T := \bigsqcup_{i=1}^{b-1} T_i$.

Now, let $k = f \circ g \circ f^{-1} \circ g^{-1}$ with $f,g \in \mathrm{Aut}_0^{\mathrm{PL}}\big(\#_{i=1}^b(S^2 \times S^5)\big)$. As $H_2(h,\mathbf{Z})$ is the identity for every element $h \in \mathrm{Aut}_0^{\mathrm{PL}}\big(\#_{i=1}^b(S^2 \times S^5)\big)$ and $S^i_{\pm}$, $i = 1,\dots,b$, both represent the same basis for $H_2(\partial W,\mathbf{Z})$, h is isotopic to a map h' which satisfies either assumption (A) or (B) below.