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By the transversality theorem ([17], IV.(2.4)), one sees that we may assume $S_i^2 \cap g_j(S^3) = \emptyset$ for all i and j .

By Corollary 3.9, the ambient isotopy class of the embedding g_k is determined by the element $\varphi_k := [g_k] \in \pi_3(W_k)$, $W_k := W \setminus \bigcup_{j \neq k} g_j(S^3)$, $k = 1, \dots, b'$. We clearly have (compare [8])

$$\pi_3(W_k) = \pi_3 \left(\underbrace{S^2 \vee \dots \vee S^2}_{b \times} \vee \underbrace{S^3 \vee \dots \vee S^3}_{(b'-1) \times} \right),$$

so that the Hilton-Milnor theorem yields

$$\pi_3(W_k) = \bigoplus_{i=1}^b \pi_3(S^2) \oplus \bigoplus_{1 \leq i < j \leq b} \pi_3(S^3) \oplus \bigoplus_{j \neq k} \pi_3(S^3).$$

Hence, we write φ_k as a tuple of integers:

$$\varphi_k = (l_i^k, i = 1, \dots, b; l_{ij}^k, 1 \leq i < j \leq b; \lambda_{kj}, j \neq k).$$

Observe that, for $j \neq k$, φ_k is mapped under the natural homomorphism

$$\pi_3(W_k) \longrightarrow H_3(W_k, \mathbf{Z}) \longrightarrow H_3(W \setminus g_j(S^3), \mathbf{Z}) (\cong \mathbf{Z})$$

to the image of the fundamental class of S^3 under g_{j*} . Thus, λ_{kj} is just the ‘usual’ linking number of the spheres $g_k(S^3)$ and $g_j(S^3)$ in W (compare [8]).

3.7 LINKS OF 5-SPHERES IN S^8

Let $\mathcal{FC}_b^{\text{PL}(C^\infty)}$ be as before, and let $C_b^{\text{PL}(C^\infty)}$ be the group of isotopy classes of piecewise linear (smooth) embeddings of b disjoint copies of S^5 into S^8 . For $b = 1$, these groups are studied in [10], [19], and [20]. A brief summary with references of results in the case $b > 1$ is contained in Section 2.6 of [11]. We will review some of this material below.

PROPOSITION 3.15. *We have $\mathcal{FC}_1^{C^\infty} \cong \mathcal{FC}_1^{\text{PL}} \cong \mathbf{Z}_2$.*

Proof. Since $\pi_5(\text{SO}(3)) \cong \mathbf{Z}_2$, the standard embedding of S^5 into S^8 with its two possible framings provides an injection of \mathbf{Z}_2 into $\mathcal{FC}_1^{\text{PL}(C^\infty)}$. By Zeeman’s unknotting theorem 3.10, the map $\mathbf{Z}_2 \longrightarrow \mathcal{FC}_1^{\text{PL}}$ is an isomorphism. As remarked in Section 2.6 of [11], $\mathcal{FC}_1^{\text{PL}}$ is isomorphic to $\mathcal{F}\vartheta$, the group of h-cobordism classes of framed submanifolds of S^8 which are homotopy 5-spheres. Moreover, by [10] and [19], there is an exact sequence

$$\dots \longrightarrow \vartheta^6 \longrightarrow \mathcal{FC}_1^{C^\infty} \longrightarrow \mathcal{F}\vartheta \longrightarrow \vartheta^5 \longrightarrow \dots$$

As the groups ϑ^5 and ϑ^6 of exotic 5- and 6-spheres are trivial [17], our claim is settled. \square

Let $L_b \subset C_b^{C^\infty}$ be the subgroup of those embeddings for which the restriction to each component is isotopic to the standard embedding. As observed in Section 2.6 of [11], Zeeman's unknotting theorem 3.10 implies that $L_b = C_b^{PL}$. The following result settles Proposition 2.3:

$$\text{COROLLARY 3.16. } \mathcal{FC}_b^{C^\infty} \cong \mathcal{FC}_b^{PL} \cong L_b \oplus \bigoplus_{i=1}^b \mathbf{Z}_2.$$

For the group L_b , Theorem 1.3 of [11] provides a fairly explicit description as an extension of abelian groups. For this, consider the b -fold wedge product $\bigvee_{i=1}^b S^2$ of 2-spheres together with its inclusion $i: \bigvee_{i=1}^b S^2 \hookrightarrow \times_{i=1}^b S^2$ into the b -fold product of 2-spheres. Finally, let $p_i: \bigvee_{i=1}^b S^2 \rightarrow S^2$ be the projection onto the i^{th} factor, $i = 1, \dots, b$. Set, for $m = 1, 2, \dots$,

$$\Lambda_{b,j}^m := \text{Ker}(\pi_m(p_j): \pi_m(\bigvee_{i=1}^b S^2) \rightarrow \pi_m(S^2)), \quad j = 1, \dots, b,$$

$$\Lambda_b^m := \bigoplus_{j=1}^b \Lambda_{b,j}^m$$

and

$$\Pi_b^m := \text{Ker}(\pi_m(i): \pi_m(\bigvee_{i=1}^b S^2) \rightarrow \bigoplus_{i=1}^b \pi_m(S^2)),$$

and define

$$w_b^m: \Lambda_b^m \rightarrow \Pi_b^{m+1}$$

on $\Lambda_{b,j}^m$ by $w_b^m(\alpha) := [\alpha, \iota_i]$. Here, $[\cdot, \cdot]$ stands for the Whitehead product inside the homotopy groups of $\bigvee_{i=1}^b S^2$ and $\iota_i: S^2 \hookrightarrow \bigvee_{i=1}^b S^2$ for the inclusion of the i^{th} factor, $i = 1, \dots, b$. Theorem 1.3 of [11] yields in our situation

THEOREM 3.17. *There is an exact sequence of abelian groups*

$$0 \rightarrow \text{Coker}(w_b^6) \rightarrow L_b \rightarrow \text{Ker}(w_b^5) \rightarrow 0.$$

We remark that the formulas of Steer [33] might be used for the explicit computation of Whitehead products and thus for the determination

of $\text{Coker}(w_b^6)$ and $\text{Ker}(w_b^5)$. The free part of L_b , e.g., can be obtained quite easily. We confine ourselves to prove the following important fact.

COROLLARY 3.18. *The group L_b has positive rank for $b \geq 2$.*

Proof. Let $L_b := \bigoplus_{l \geq 1} L_{b,l}$ be the free graded Lie algebra with $L_{b,1} := \bigoplus_{i=1}^b \mathbf{Z} \cdot e_i$. For $l = 2, 3, \dots$, let $e_1^l, \dots, e_{d_l}^l$ be a basis for $L_{b,l}$ consisting of iterated commutators of the e_i . By assigning ι_i to e_i , every iterated commutator $c \in L_{b,l}$ in the e_i defines an element $\alpha(c) \in \pi_{l+1}(\bigvee_{i=1}^b S^2)$.

To settle our claim, it is certainly sufficient to show that $\text{Coker}(w_b^6)$ has positive rank. Now, by the Hilton-Milnor theorem

$$\Pi_b^7 \cong \bigoplus_{l=3}^7 \bigoplus_{k=1}^{d_{l-1}} \pi_7(S^l) \cdot \alpha(e_k^{l-1}).$$

Note that $\pi_7(S^l)$ is finite for $l \notin \{4, 7\}$ (see [32] and [35] for the explicit description of those groups). The Hopf fibration $S^7 \rightarrow S^4$ [32], on the other hand, yields a decomposition $\pi_7(S^4) \cong \pi_6(S^3) \oplus \pi_7(S^7) \cong \mathbf{Z}_{12} \oplus \mathbf{Z}$. Therefore, it will suffice to show that the free part of Λ_b^6 is mapped to $\bigoplus_{j=1}^{d_6} \pi_7(S^7) \cdot \alpha(e_j^6)$. For $j = 1, \dots, b$, we have

$$\Lambda_{b,j}^6 \cong \bigoplus_{i \neq j} \pi_6(S^2) \cdot \iota_i \oplus \bigoplus_{l=3}^6 \bigoplus_{k=1}^{d_{l-1}} \pi_6(S^l) \cdot \alpha(e_k^{l-1}).$$

The group $\pi_6(S^l)$ is finite for $l < 6$, and we obviously have $[\alpha(e_k^5), \iota_j] = \alpha([e_k^5, e_j])$. If we expand the commutator $[e_k^5, e_j]$ in the basis $e_1^6, \dots, e_{d_6}^6$, we find an expansion for $[\alpha(e_k^6), \iota_j]$ in terms of the $\alpha(e_k^6)$. \square

COROLLARY 3.19. *The set of $\text{GL}_b(\mathbf{Z})$ -equivalence classes of elements in L_b is infinite for $b \geq 2$.*

Proof. We have seen that the $\text{GL}_b(\mathbf{Z})$ -set $L_{b,3}$ is contained in the $\text{GL}_b(\mathbf{Z})$ -set L_b . The $\text{GL}_b(\mathbf{Z})$ -action on $L_{b,3}$ originates from a homomorphism $\text{GL}_b(\mathbf{Z}) \rightarrow \text{GL}(L_{b,3}) := \text{Aut}_{\mathbf{Z}}(L_{b,3})$. In particular, any matrix $g \in \text{GL}_b(\mathbf{Z})$ preserves the absolute value of the determinant of any d_3 elements in $L_{b,3}$. This implies, for instance, that $a \cdot e_1^3$ and $b \cdot e_1^3$ cannot lie in the same $\text{GL}_b(\mathbf{Z})$ -orbit, if $0 \leq a < b$. \square