

Zeitschrift:	L'Enseignement Mathématique
Herausgeber:	Commission Internationale de l'Enseignement Mathématique
Band:	48 (2002)
Heft:	3-4: L'ENSEIGNEMENT MATHÉMATIQUE
Artikel:	ON THE CLASSIFICATION OF CERTAIN PIECEWISE LINEAR AND DIFFERENTIABLE MANIFOLDS IN DIMENSION EIGHT AND AUTOMORPHISMS OF $\sharp_{i=1}^b (S^2 \times S^5)$
Autor:	Schmitt, Alexander
Kapitel:	3.7 Links of 5-spheres in S^8
DOI:	https://doi.org/10.5169/seals-66077

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 05.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

By the transversality theorem ([17], IV.(2.4)), one sees that we may assume $S_i^2 \cap g_j(S^3) = \emptyset$ for all i and j .

By Corollary 3.9, the ambient isotopy class of the embedding g_k is determined by the element $\varphi_k := [g_k] \in \pi_3(W_k)$, $W_k := W \setminus \bigcup_{j \neq k} g_j(S^3)$, $k = 1, \dots, b'$. We clearly have (compare [8])

$$\pi_3(W_k) = \pi_3\left(\underbrace{S^2 \vee \cdots \vee S^2}_{b \times} \vee \underbrace{S^3 \vee \cdots \vee S^3}_{(b'-1) \times}\right),$$

so that the Hilton-Milnor theorem yields

$$\pi_3(W_k) = \bigoplus_{i=1}^b \pi_3(S^2) \oplus \bigoplus_{1 \leq i < j \leq b} \pi_3(S^3) \oplus \bigoplus_{j \neq k} \pi_3(S^3).$$

Hence, we write φ_k as a tuple of integers:

$$\varphi_k = (l_i^k, i = 1, \dots, b; \quad l_{ij}^k, 1 \leq i < j \leq b; \quad \lambda_{kj}, j \neq k).$$

Observe that, for $j \neq k$, φ_k is mapped under the natural homomorphism

$$\pi_3(W_k) \longrightarrow H_3(W_k, \mathbf{Z}) \longrightarrow H_3(W \setminus g_j(S^3), \mathbf{Z}) (\cong \mathbf{Z})$$

to the image of the fundamental class of S^3 under g_{j*} . Thus, λ_{kj} is just the ‘usual’ linking number of the spheres $g_k(S^3)$ and $g_j(S^3)$ in W (compare [8]).

3.7 LINKS OF 5-SPHERES IN S^8

Let $\mathcal{FC}_b^{\text{PL}(C^\infty)}$ be as before, and let $\mathcal{C}_b^{\text{PL}(C^\infty)}$ be the group of isotopy classes of piecewise linear (smooth) embeddings of b disjoint copies of S^5 into S^8 . For $b = 1$, these groups are studied in [10], [19], and [20]. A brief summary with references of results in the case $b > 1$ is contained in Section 2.6 of [11]. We will review some of this material below.

PROPOSITION 3.15. *We have $\mathcal{FC}_1^{C^\infty} \cong \mathcal{FC}_1^{\text{PL}} \cong \mathbf{Z}_2$.*

Proof. Since $\pi_5(\text{SO}(3)) \cong \mathbf{Z}_2$, the standard embedding of S^5 into S^8 with its two possible framings provides an injection of \mathbf{Z}_2 into $\mathcal{FC}_1^{\text{PL}(C^\infty)}$. By Zeeman’s unknotting theorem 3.10, the map $\mathbf{Z}_2 \longrightarrow \mathcal{FC}_1^{\text{PL}}$ is an isomorphism. As remarked in Section 2.6 of [11], $\mathcal{FC}_1^{\text{PL}}$ is isomorphic to $\mathcal{F}\vartheta$, the group of h-cobordism classes of framed submanifolds of S^8 which are homotopy 5-spheres. Moreover, by [10] and [19], there is an exact sequence

$$\cdots \longrightarrow \vartheta^6 \longrightarrow \mathcal{FC}_1^{C^\infty} \longrightarrow \mathcal{F}\vartheta \longrightarrow \vartheta^5 \longrightarrow \cdots.$$

As the groups ϑ^5 and ϑ^6 of exotic 5- and 6-spheres are trivial [17], our claim is settled. \square

Let $L_b \subset \mathcal{C}_b^{\mathcal{C}^\infty}$ be the subgroup of those embeddings for which the restriction to each component is isotopic to the standard embedding. As observed in Section 2.6 of [11], Zeeman's unknotting theorem 3.10 implies that $L_b = \mathcal{C}_b^{\text{PL}}$. The following result settles Proposition 2.3:

COROLLARY 3.16. $\mathcal{FC}_b^{\mathcal{C}^\infty} \cong \mathcal{FC}_b^{\text{PL}} \cong L_b \oplus \bigoplus_{i=1}^b \mathbf{Z}_2$.

For the group L_b , Theorem 1.3 of [11] provides a fairly explicit description as an extension of abelian groups. For this, consider the b -fold wedge product $\bigvee_{i=1}^b S^2$ of 2-spheres together with its inclusion $i: \bigvee_{i=1}^b S^2 \hookrightarrow X_{i=1}^b S^2$ into the b -fold product of 2-spheres. Finally, let $p_i: \bigvee_{i=1}^b S^2 \rightarrow S^2$ be the projection onto the i^{th} factor, $i = 1, \dots, b$. Set, for $m = 1, 2, \dots$,

$$\Lambda_{b,j}^m := \text{Ker}(\pi_m(p_j): \pi_m(\bigvee_{i=1}^b S^2) \rightarrow \pi_m(S^2)), \quad j = 1, \dots, b,$$

$$\Lambda_b^m := \bigoplus_{j=1}^b \Lambda_{b,j}^m$$

and

$$\Pi_b^m := \text{Ker}(\pi_m(i): \pi_m(\bigvee_{i=1}^b S^2) \rightarrow \bigoplus_{i=1}^b \pi_m(S^2)),$$

and define

$$w_b^m: \Lambda_b^m \rightarrow \Pi_b^{m+1}$$

on $\Lambda_{b,j}^m$ by $w_b^m(\alpha) := [\alpha, \iota_i]$. Here, $[., .]$ stands for the Whitehead product inside the homotopy groups of $\bigvee_{i=1}^b S^2$ and $\iota_i: S^2 \hookrightarrow \bigvee_{i=1}^b S^2$ for the inclusion of the i^{th} factor, $i = 1, \dots, b$. Theorem 1.3 of [11] yields in our situation

THEOREM 3.17. *There is an exact sequence of abelian groups*

$$0 \rightarrow \text{Coker}(w_b^6) \rightarrow L_b \rightarrow \text{Ker}(w_b^5) \rightarrow 0.$$

We remark that the formulas of Steer [33] might be used for the explicit computation of Whitehead products and thus for the determination

of $\text{Coker}(w_b^6)$ and $\text{Ker}(w_b^5)$. The free part of \mathbf{L}_b , e.g., can be obtained quite easily. We confine ourselves to prove the following important fact.

COROLLARY 3.18. *The group \mathbf{L}_b has positive rank for $b \geq 2$.*

Proof. Let $\mathbf{L}_b := \bigoplus_{l \geq 1} \mathbf{L}_{b,l}$ be the free graded Lie algebra with $\mathbf{L}_{b,1} := \bigoplus_{i=1}^b \mathbf{Z} \cdot e_i$. For $l = 2, 3, \dots$, let $e_1^l, \dots, e_{d_l}^l$ be a basis for $\mathbf{L}_{b,l}$ consisting of iterated commutators of the e_i . By assigning ι_i to e_i , every iterated commutator $c \in \mathbf{L}_{b,l}$ in the e_i defines an element $\alpha(c) \in \pi_{l+1}(\bigvee_{i=1}^b S^2)$.

To settle our claim, it is certainly sufficient to show that $\text{Coker}(w_b^6)$ has positive rank. Now, by the Hilton-Milnor theorem

$$\Pi_b^7 \cong \bigoplus_{l=3}^7 \bigoplus_{k=1}^{d_{l-1}} \pi_7(S^l) \cdot \alpha(e_k^{l-1}).$$

Note that $\pi_7(S^l)$ is finite for $l \notin \{4, 7\}$ (see [32] and [35] for the explicit description of those groups). The Hopf fibration $S^7 \rightarrow S^4$ [32], on the other hand, yields a decomposition $\pi_7(S^4) \cong \pi_6(S^3) \oplus \pi_7(S^7) \cong \mathbf{Z}_{12} \oplus \mathbf{Z}$. Therefore, it will suffice to show that the free part of Λ_b^6 is mapped to $\bigoplus_{j=1}^{d_6} \pi_7(S^7) \cdot \alpha(e_j^6)$. For $j = 1, \dots, b$, we have

$$\Lambda_{b,j}^6 \cong \bigoplus_{i \neq j} \pi_6(S^2) \cdot \iota_i \oplus \bigoplus_{l=3}^6 \bigoplus_{k=1}^{d_{l-1}} \pi_6(S^l) \cdot \alpha(e_k^{l-1}).$$

The group $\pi_6(S^l)$ is finite for $l < 6$, and we obviously have $[\alpha(e_k^5), \iota_j] = \alpha([e_k^5, e_j])$. If we expand the commutator $[e_k^5, e_j]$ in the basis $e_1^6, \dots, e_{d_6}^6$, we find an expansion for $[\alpha(e_k^6), \iota_j]$ in terms of the $\alpha(e_k^6)$. \square

COROLLARY 3.19. *The set of $\text{GL}_b(\mathbf{Z})$ -equivalence classes of elements in \mathbf{L}_b is infinite for $b \geq 2$.*

Proof. We have seen that the $\text{GL}_b(\mathbf{Z})$ -set $\mathbf{L}_{b,3}$ is contained in the $\text{GL}_b(\mathbf{Z})$ -set \mathbf{L}_b . The $\text{GL}_b(\mathbf{Z})$ -action on $\mathbf{L}_{b,3}$ originates from a homomorphism $\text{GL}_b(\mathbf{Z}) \rightarrow \text{GL}(\mathbf{L}_{b,3}) := \text{Aut}_{\mathbf{Z}}(\mathbf{L}_{b,3})$. In particular, any matrix $g \in \text{GL}_b(\mathbf{Z})$ preserves the absolute value of the determinant of any d_3 elements in $\mathbf{L}_{b,3}$. This implies, for instance, that $a \cdot e_1^3$ and $b \cdot e_1^3$ cannot lie in the same $\text{GL}_b(\mathbf{Z})$ -orbit, if $0 \leq a < b$. \square