2.2 Manifolds with trivial cup form δ_X

In addition to describing the explicit geometric meaning of the system of invariants of an E-manifold X with $w_2(X) = 0$, we will describe those manifolds X for which the cup form δ_X vanishes.

For any $b > 0$, let $\mathcal{FC}_b^{\mathrm{PL}(\mathcal{C}^\infty)}$ be the group of isotopy classes of piecewise linear (smooth) embeddings of b disjoint copies of $S^5 \times D^3$ into S^8. The following result will be established in Section 3.7.

Proposition 2.3. $\mathrm{FL}_b := \mathcal{FC}_b^{\mathcal{C}^\infty} \cong \mathcal{FC}_b^{\mathrm{PL}}$.

Given an element $[l] \in \mathrm{FL}_b$, we can perform surgery along the link l and get a smooth based E-manifold $(X(l), \underline{x}(l))$ with $w_2(X) = 0$ and $b_4(X) = 0$, which is well defined up to smooth isomorphy of based manifolds.

We will also use the following notation: Fix a pair $(\gamma, p) \in Z(0, b')$ which satisfies relation (2) (and (3)) and denote by $\mathfrak{I}^{\mathrm{PL}(\mathcal{C}^\infty)}(b, \gamma, p)$ the set of isomorphy classes of based piecewise linear (smooth) E-manifolds $(X, \underline{x}, \underline{y})$ with $w_2(X) = 0$, $b_2(X) = b$, $\gamma_X = \gamma$, and $p_1(X) = p$. Pick a three-connected piecewise linear (smooth) based E-manifold $(X_0, \underline{y}_0)$ with $\gamma_{X_0} = \gamma$ and $p_1(X_0) = p$. In the smooth case, let $\vartheta^8 \cong \mathbf{Z}_2$ [17] be the group of exotic 8-spheres, and set $\vartheta(X_0) := \vartheta^8$, if X_0 is not diffeomorphic to $X_0 \# \Sigma$, Σ a generator for ϑ^8, and $\vartheta(X_0) := \{[S^8]\} \subset \vartheta^8$ otherwise. Now, we define maps

$$
\begin{aligned}
K^{\mathrm{PL}}(b, \gamma, p) \colon \mathrm{FL}_b &\longrightarrow \mathfrak{I}^{\mathrm{PL}}(b, \gamma, p) \\
[l] &\longmapsto \left[X(l) \# X_0, \underline{x}(l), \underline{y}_0\right]
\end{aligned}
$$

and

$$
\begin{aligned}
K_{X_0}^{\mathcal{C}^\infty}(b, \gamma, p) \colon \mathrm{FL}_b \oplus \vartheta(X_0) &\longrightarrow \mathfrak{I}^{\mathcal{C}^\infty}(b, \gamma, p) \\
([l], [\Sigma]) &\longmapsto \left[X(l) \# X_0 \# \Sigma, \underline{x}(l), \underline{y}_0\right] .
\end{aligned}
$$

In $\mathfrak{I}^{\mathrm{PL}(\mathcal{C}^\infty)}(b, \gamma, p)$, we mark the class $\left[\left(\#_{i=1}^b (S^2 \times S^6)\right) \# X_0, \underline{x}, \underline{y}_0\right]$, where $\underline{x}$ comes from the natural basis of $H^2(\#_{i=1}^b (S^2 \times S^6), \mathbf{Z})$. Then our main result is the following

THEOREM 2.4. i) *For every* $b > 0$ *and every pair* (γ, p) *which satisfies the relation* (2),

$$\begin{array}{ccccccc} \{1\} & \longrightarrow & \mathrm{FL}_b & \stackrel{K^{\mathrm{PL}}(b,\gamma,p)}{\longrightarrow} & \mathfrak{I}^{\mathrm{PL}}(b,\gamma,\varphi) & \longrightarrow & \mathrm{Hom}(S^2\mathbf{Z}^b, Z^{b'}) \\ 1 & \longmapsto & [\textit{trivial link}] & & [X,\underline{x},\underline{y}] & \longmapsto & \delta_X \end{array}$$

is an exact sequence of pointed sets.

ii) *For every* $b > 0$ *and every pair* (γ, p) *which satisfies the relations* (2) *and* (3),

$$\begin{array}{ccccccc} \{1\} & \longrightarrow & \mathrm{FL}_b \oplus \vartheta(X_0) & \stackrel{K^{\mathcal{C}^\infty}_{X_0}(b,\gamma,p)}{\longrightarrow} & \mathfrak{I}^{\mathcal{C}^\infty}(b,\gamma,\varphi) & \longrightarrow & \mathrm{Hom}(S^2\mathbf{Z}^b, Z^{b'}) \\ 1 & \longmapsto & [\textit{trivial link}] & & [X,\underline{x},\underline{y}] & \longmapsto & \delta_X \end{array}$$

is an exact sequence of pointed sets.

The proof will be given in Sections 4.2 and 4.3.

REMARK 2.5. i) In the PL setting, we will show that the inclusion of FL_b into $\mathfrak{I}^{\mathrm{PL}}(b,\gamma,\varphi)$ extends to an action of FL_b on $\mathfrak{I}^{\mathrm{PL}}(b,\gamma,\varphi)$, such that the orbits are the fibres of the map $[X,\underline{x},\underline{y}] \longmapsto \delta_X$.

ii) On all the sets occurring in Theorem 2.4 there are natural $(\mathrm{GL}_b(\mathbf{Z}) \times \mathrm{GL}_{b'}(\mathbf{Z}))$-actions, and the maps are equivariant for these actions. Therefore, by forming the equivalence classes with respect to these actions, we get the classification of E-manifolds with vanishing second Stiefel-Whitney class up to orientation preserving piecewise linear (smooth) isomorphy.

iii) We will discuss in Section 3.7 the structure of the group FL_b. It turns out that it is finite if and only if $b = 1$. It follows easily that the set of $\mathrm{GL}_b(\mathbf{Z})$-equivalence classes in FL_b is infinite for $b \geq 2$. Thus, the cohomology ring and the characteristic classes classify E-manifolds of dimension eight up to finite indeterminacy only if the second Betti number is at most one.

The starting point of our proof of the above results will be the theory of minimal handle decompositions of Smale which states that X can be obtained from D^8 by first attaching $b_2(X)$ 2-handles, then $b_4(X)$ 4-handles, then $b_2(X)$ 6-handles and finally one 8-handle. At each step, the attachment will be determined by the isotopy class of a certain framed link in a 7-manifold, and we will first explain how to read off the isotopy class of the attaching links for the 2- and 4-handles from the invariants.