

Zeitschrift:	L'Enseignement Mathématique
Herausgeber:	Commission Internationale de l'Enseignement Mathématique
Band:	48 (2002)
Heft:	3-4: L'ENSEIGNEMENT MATHÉMATIQUE
Artikel:	ON THE CLASSIFICATION OF CERTAIN PIECEWISE LINEAR AND DIFFERENTIABLE MANIFOLDS IN DIMENSION EIGHT AND AUTOMORPHISMS OF $\sharp_{i=1}^b (S^2 \times S^5)$
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Kapitel:	2.1 The classical invariants
DOI:	https://doi.org/10.5169/seals-66077

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and $f^*(\underline{y}') = \underline{y}$. We denote by $\mathcal{J}^{\text{PL}(\mathcal{C}^\infty)}(b, b')$ the set of isomorphy classes of piecewise linear (smooth) based E-manifolds $(X, \underline{x}, \underline{y})$ of dimension eight with vanishing second Stiefel-Whitney class, $b_2(X) = b$, and $b_4(X) = b'$.

2.1 THE CLASSICAL INVARIANTS

In the terminology of [24], the classical invariants of an E-manifold consist of its cohomology ring, the Stiefel-Whitney classes, the Wu classes, the Pontrjagin classes, the Euler class, the Steenrod squares, the reduced Steenrod powers, and the Pontrjagin powers. For an eight-dimensional E-manifold X with vanishing second Stiefel-Whitney class, the main result of [24] states that the classical invariants are fully determined by the following invariants :

1. The cup product map

$$\begin{aligned}\delta_X: S^2 H^2(X, \mathbf{Z}) &\longrightarrow H^4(X, \mathbf{Z}) \\ x \otimes x' &\longmapsto x \cup x'.\end{aligned}$$

2. The intersection form

$$\begin{aligned}\gamma_X: S^2 H^4(X, \mathbf{Z}) &\longrightarrow \mathbf{Z} \\ y \otimes y' &\longmapsto (y \cup y')[X].\end{aligned}$$

Here, $[X] \in H_8(X, \mathbf{Z})$ is the fundamental class determined by the orientation.

3. The first Pontrjagin class $p_1(X) \in H^4(X, \mathbf{Q})$.

REMARK 2.1. The above invariants are not independent. By associativity of the cohomology ring, the following relation holds

$$(1) \quad \delta_X^*(\gamma_X) \in S^4 H^2(X, \mathbf{Z})^\vee,$$

i.e.,

$$\gamma_X(\delta_X(x_1 \otimes x_2) \otimes \delta_X(x_3 \otimes x_4)) = \gamma_X(\delta_X(x_1 \otimes x_3) \otimes \delta_X(x_2 \otimes x_4)),$$

for all $x_1, x_2, x_3, x_4 \in H^2(X, \mathbf{Z})$. Furthermore,

PROPOSITION ([24], Prop. A.7 or Cor. 3.14 below). *For every element $y \in H^4(X, \mathbf{Z})$ we have*

$$(2) \quad p_1(X)y \equiv 2y^2 \pmod{4}.$$

Note that this implies $p_1(X) \in H^4(X, \mathbf{Z})$. If, in addition, X is differentiable then for every integral lift $W \in H^2(X, \mathbf{Z})$ of $w_2(X)$ one has

$$(3) \quad 3p_1(X)^2 - 14p_1(X)W^2 + 7W^4 \equiv 12 \operatorname{Sign}(\gamma_X) \pmod{2688}.$$

Müller has also shown [24] that these relations imply all the other relations among the classical invariants of X . Conversely, a piecewise linear manifold X whose invariants obey relation (3) admits a differentiable structure [18], [24].

We are led to the following algebraic concept: A *system of invariants of type (b, b')* is a triple (δ, γ, p) , consisting of

- a homomorphism $\delta: S^2 \mathbf{Z}^{\oplus b} \longrightarrow \mathbf{Z}^{\oplus b'}$,
- a unimodular symmetric bilinear form $\gamma: S^2 \mathbf{Z}^{\oplus b'} \longrightarrow \mathbf{Z}$, and
- an element $p \in \mathbf{Z}^{\oplus b'}$.

We denote by $Z(b, b')$ the set of systems of invariants of type (b, b') .

Now, let $(X, \underline{x}, \underline{y})$ be a based eight-dimensional E-manifold. This defines a set of invariants $Z_{(X, \underline{x}, \underline{y})} := (\delta_X, \gamma_X, p_1(X))$ of type $(b_2(X), b_4(X))$. Thus, we have natural maps

$$\begin{aligned} Z^{\text{PL}(\mathcal{C}^\infty)}(b, b') &: \mathfrak{I}^{\text{PL}(\mathcal{C}^\infty)}(b, b') \longrightarrow Z(b, b') \\ &[\underline{X}, \underline{x}, \underline{y}] \longmapsto Z_{(X, \underline{x}, \underline{y})}. \end{aligned}$$

It will be the concern of our paper to understand the maps $Z^{\text{PL}(\mathcal{C}^\infty)}$ as well as possible. The first result can be easily derived from Wall's work [36] and deals with the case $b = 0$. It will be proved in detail in Section 4.1.

THEOREM 2.2. i) *The map $Z^{\text{PL}}(0, b')$ is injective. Its image consists precisely of those elements which satisfy the relations (1) and (2).*

ii) *Given two smooth based E-manifolds (X, \underline{y}) and (X', \underline{y}') with $b_2(X) = 0 = b_2(X')$ and $Z_{(X, \underline{y})} = Z_{(X', \underline{y}')}$, there exists an exotic 8-sphere Σ such that $(X \# \Sigma, \underline{y})$ and (X', \underline{y}') are smoothly isomorphic. In particular, the fibres of $Z^{\mathcal{C}^\infty}$ have cardinality at most two. The image of $Z^{\mathcal{C}^\infty}$ consists exactly of those elements which satisfy the relations (1), (2), and (3).*