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We shall give an example of an homology 4-sphere whose group has deficiency < 0. Thus none of the implications "G is an homology 3-sphere group"  $\Rightarrow$  "G is finitely presentable, perfect and def(G) = 0"  $\Rightarrow$  "G is an homology 4-sphere group"  $\Rightarrow$  "G is finitely presentable and superperfect" can be reversed.

A similar outcome was known for knots by the late 1970s. (Namely, none of the implications "G is a 1-knot group"  $\Rightarrow$  "G is a high dimensional knot group and def(G) = 1"  $\Rightarrow$  "G is a 2-knot group"  $\Rightarrow$  "G is a high dimensional knot group" can be reversed [Fo62, Ke65, Fa75]). The issue considered here was raised by Plotnick, who suggested a possible example [Pl82]. (See also [Be02]). We use a related construction, but our example is different, and we do not know whether Plotnick's candidates indeed have negative deficiency.

The construction starts with a 2-knot  $K: S^2 \to S^4$  and an homology 4-sphere  $\Sigma$ . Let M be the closed 4-manifold obtained by surgery on K, and let  $N = M \sharp \Sigma$ . Let  $G = \pi_1(M)$  and  $H = \pi_1(\Sigma)$ . (Thus G is the group of the knot K.) Let  $t \in G$  represent a generator of  $G/G' \cong Z$ , and let  $h \in H$ . The conjugacy class of  $th^{-1} \in \pi_1(N) \cong G * H$  is represented by an unique isotopy class of embeddings of  $S^1$  in N. Surgery on such an embedding gives an homology 4-sphere P, with group  $\pi = \pi_1(P) = (G * H)/\langle\langle th^{-1} \rangle\rangle$ .

Let  $\rho = \langle \langle G' \rangle \rangle_{\pi}$  be the normal closure of the image of G' in  $\pi$ . Then  $\pi/\rho \cong H$ , and so  $\pi$  is the semidirect product  $\rho \rtimes H$ . Let  $\Gamma = \mathbf{Z}[H]$  and let  $I = \operatorname{Ker}(\varepsilon \colon \Gamma \to \mathbf{Z})$  be the augmentation ideal of H. Since H is finitely presentable I has a resolution  $C_*$  by free left  $\Gamma$ -modules which are finitely generated in degrees  $\leq 2$ . Let  $B = H_1(\pi; \Gamma) \cong \rho/\rho'$ . Then B is a left  $\Gamma$ -module and there is an exact sequence  $0 \to B \to A \to I \to 0$ , in which  $A = H_1(\pi, 1; \Gamma)$  is a relative homology group [Cr61]. Evaluating the Jacobian matrix associated to a presentation for  $\pi$  via the natural epimorphism from  $\mathbf{Z}[\pi]$  to  $\Gamma$  gives a presentation matrix for A as a module (see [Cr61] or [Fo62]). Thus there is an exact sequence  $D_* : \cdots \to \Gamma^m \to \Gamma^n \to A \to 0$ , where  $n-m = \operatorname{def}(\pi)$ . A mapping cone construction leads to an exact sequence of the form  $C_2 \oplus D_1 \to C_1 \oplus D_0 \to B \oplus C_0 \to 0$  and hence to a presentation for B of the form  $C_2 \oplus D_1 \oplus C_0 \to C_1 \oplus D_0 \to B$ .

Now let K be the 2-twist spin of the trefoil knot, with group  $G = \langle x, s \mid x^3 = 1, sxs^{-1} = x^{-1} \rangle$ , and let H be the Higman group with presentation  $\langle a, b, c, d \mid bab^{-1} = a^2, cbc^{-1} = b^2, dcd^{-1} = c^2, ada^{-1} = d^2 \rangle$ . Then H is perfect and def(H) = 0, so there is an homology 4-sphere  $\Sigma$  with group H. Moreover H has cohomological dimension 2 [DV73], and so

there is a short exact sequence  $0 \to \Gamma^4 \to \Gamma^4 \to I \to 0$ . Let t = s and h = a. Then  $\pi = (G*H)/\langle\langle sa^{-1}\rangle\rangle$  has a presentation of deficiency -1, and  $B \cong \Gamma/\Gamma(3, a+1)$ . Since  $B \cong \Gamma \otimes_{\Lambda} (\Lambda/\Lambda(3, a+1))$ , where  $\Lambda = \mathbb{Z}[a, a^{-1}]$ , there is an exact sequence

$$0 \to \Gamma \xrightarrow{(3,a+1)} \Gamma^2 \xrightarrow{\begin{pmatrix} a+1 \\ -3 \end{pmatrix}} \Gamma \to B \to 0.$$

Suppose that  $\pi$  has deficiency 0. Then B has deficiency 0 as a left  $\Gamma$ -module, by the general argument above. Hence there is an exact sequence

$$0 \to L \to \Gamma^p \to \Gamma^p \to B \to 0$$
.

Schanuel's Lemma gives an isomorphism  $\Gamma^{1+p+1}\cong L\oplus \Gamma^{p+2}$ , on comparing these two resolutions of B. The endomorphism of  $\Gamma^{p+2}$  given by projection onto the second summand is an automorphism, by a theorem of Kaplansky (see page 122 of [Ka69]). Hence L=0 and so B has a short free resolution. In particular,  $\operatorname{Tor}_2^{\Gamma}(R,B)=0$  for any right  $\Gamma$ -module R. But it is easily verified that if  $\bar{B}\cong \Gamma/(3,a+1)\Gamma$  is the conjugate right  $\Gamma$ -module then  $\operatorname{Tor}_2^{\Gamma}(\bar{B},B)\neq 0$ . Thus our assumption was wrong, and  $\operatorname{def}(\pi)=-1<0$ .

The group of the 2-twist spin of the trefoil knot is the simplest 2-knot group with deficiency 0 [Fo62]. Levine showed that the group of the sum of r copies of this knot has deficiency 1-r [Le78]. If we use this sum in our construction above  $\pi$  now has a presentation of deficiency -r and  $B \cong (\Gamma/\Gamma(3,a+1))^r$ , so there is an exact sequence

$$0 \to \Gamma^r \to \Gamma^{2r} \to \Gamma^r \to B \to 0$$
.

Is  $def(\pi) = -r$ ?

Is there a *finite* homology 4-sphere group of negative deficiency? Our example above is "very infinite" in the sense that the Higman group H has no finite quotients, and therefore no finite-dimensional representations over any field [Hi51]. The simplest candidate to consider is perhaps the semidirect product of  $SL(2, \mathbf{F}_5)$  with the normal subgroup  $\mathbf{F}_5^2$ , and with the natural action of  $SL(2, \mathbf{F}_5)$  on  $\mathbf{F}_5^2$ . (This semidirect product has a presentation with 3 generators and 5 relations, is superperfect, and has order 3000. I do not know whether it is the group of an homology 4-sphere, nor whether it has deficiency 0.)