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We shall give an example of an homology 4-sphere whose group has deficiency < 0 . Thus none of the implications “ G is an homology 3-sphere group” \Rightarrow “ G is finitely presentable, perfect and $\text{def}(G) = 0$ ” \Rightarrow “ G is an homology 4-sphere group” \Rightarrow “ G is finitely presentable and superperfect” can be reversed.

A similar outcome was known for knots by the late 1970s. (Namely, none of the implications “ G is a 1-knot group” \Rightarrow “ G is a high dimensional knot group and $\text{def}(G) = 1$ ” \Rightarrow “ G is a 2-knot group” \Rightarrow “ G is a high dimensional knot group” can be reversed [Fo62, Ke65, Fa75]). The issue considered here was raised by Plotnick, who suggested a possible example [Pl82]. (See also [Be02]). We use a related construction, but our example is different, and we do not know whether Plotnick’s candidates indeed have negative deficiency.

The construction starts with a 2-knot $K: S^2 \rightarrow S^4$ and an homology 4-sphere Σ . Let M be the closed 4-manifold obtained by surgery on K , and let $N = M \# \Sigma$. Let $G = \pi_1(M)$ and $H = \pi_1(\Sigma)$. (Thus G is the group of the knot K .) Let $t \in G$ represent a generator of $G/G' \cong \mathbb{Z}$, and let $h \in H$. The conjugacy class of $th^{-1} \in \pi_1(N) \cong G * H$ is represented by an unique isotopy class of embeddings of S^1 in N . Surgery on such an embedding gives an homology 4-sphere P , with group $\pi = \pi_1(P) = (G * H) / \langle\langle th^{-1} \rangle\rangle$.

Let $\rho = \langle\langle G' \rangle\rangle_\pi$ be the normal closure of the image of G' in π . Then $\pi/\rho \cong H$, and so π is the semidirect product $\rho \rtimes H$. Let $\Gamma = \mathbb{Z}[H]$ and let $I = \text{Ker}(\varepsilon: \Gamma \rightarrow \mathbb{Z})$ be the augmentation ideal of H . Since H is finitely presentable I has a resolution C_* by free left Γ -modules which are finitely generated in degrees ≤ 2 . Let $B = H_1(\pi; \Gamma) \cong \rho/\rho'$. Then B is a left Γ -module and there is an exact sequence $0 \rightarrow B \rightarrow A \rightarrow I \rightarrow 0$, in which $A = H_1(\pi, 1; \Gamma)$ is a relative homology group [Cr61]. Evaluating the Jacobian matrix associated to a presentation for π via the natural epimorphism from $\mathbb{Z}[\pi]$ to Γ gives a presentation matrix for A as a module (see [Cr61] or [Fo62]). Thus there is an exact sequence $D_*: \dots \rightarrow \Gamma^m \rightarrow \Gamma^n \rightarrow A \rightarrow 0$, where $n - m = \text{def}(\pi)$. A mapping cone construction leads to an exact sequence of the form $C_2 \oplus D_1 \rightarrow C_1 \oplus D_0 \rightarrow B \oplus C_0 \rightarrow 0$ and hence to a presentation for B of the form $C_2 \oplus D_1 \oplus C_0 \rightarrow C_1 \oplus D_0 \rightarrow B$.

Now let K be the 2-twist spin of the trefoil knot, with group $G = \langle x, s \mid x^3 = 1, sxs^{-1} = x^{-1} \rangle$, and let H be the Higman group with presentation $\langle a, b, c, d \mid bab^{-1} = a^2, cbc^{-1} = b^2, dcd^{-1} = c^2, ada^{-1} = d^2 \rangle$. Then H is perfect and $\text{def}(H) = 0$, so there is an homology 4-sphere Σ with group H . Moreover H has cohomological dimension 2 [DV73], and so

there is a short exact sequence $0 \rightarrow \Gamma^4 \rightarrow \Gamma^4 \rightarrow I \rightarrow 0$. Let $t = s$ and $h = a$. Then $\pi = (G * H) / \langle\langle sa^{-1} \rangle\rangle$ has a presentation of deficiency -1 , and $B \cong \Gamma / \Gamma(3, a+1)$. Since $B \cong \Gamma \otimes_{\Lambda} (\Lambda / \Lambda(3, a+1))$, where $\Lambda = \mathbb{Z}[a, a^{-1}]$, there is an exact sequence

$$0 \rightarrow \Gamma \xrightarrow{(3, a+1)} \Gamma^2 \xrightarrow{\begin{pmatrix} a+1 \\ -3 \end{pmatrix}} \Gamma \rightarrow B \rightarrow 0.$$

Suppose that π has deficiency 0. Then B has deficiency 0 as a left Γ -module, by the general argument above. Hence there is an exact sequence

$$0 \rightarrow L \rightarrow \Gamma^p \rightarrow \Gamma^p \rightarrow B \rightarrow 0.$$

Schanuel's Lemma gives an isomorphism $\Gamma^{1+p+1} \cong L \oplus \Gamma^{p+2}$, on comparing these two resolutions of B . The endomorphism of Γ^{p+2} given by projection onto the second summand is an automorphism, by a theorem of Kaplansky (see page 122 of [Ka69]). Hence $L = 0$ and so B has a short free resolution. In particular, $\text{Tor}_2^{\Gamma}(R, B) = 0$ for any right Γ -module R . But it is easily verified that if $\bar{B} \cong \Gamma / (3, a+1)\Gamma$ is the conjugate right Γ -module then $\text{Tor}_2^{\Gamma}(\bar{B}, B) \neq 0$. Thus our assumption was wrong, and $\text{def}(\pi) = -1 < 0$.

The group of the 2-twist spin of the trefoil knot is the simplest 2-knot group with deficiency 0 [Fo62]. Levine showed that the group of the sum of r copies of this knot has deficiency $1 - r$ [Le78]. If we use this sum in our construction above π now has a presentation of deficiency $-r$ and $B \cong (\Gamma / \Gamma(3, a+1))^r$, so there is an exact sequence

$$0 \rightarrow \Gamma^r \rightarrow \Gamma^{2r} \rightarrow \Gamma^r \rightarrow B \rightarrow 0.$$

Is $\text{def}(\pi) = -r$?

Is there a *finite* homology 4-sphere group of negative deficiency? Our example above is "very infinite" in the sense that the Higman group H has no finite quotients, and therefore no finite-dimensional representations over any field [Hi51]. The simplest candidate to consider is perhaps the semidirect product of $\text{SL}(2, \mathbb{F}_5)$ with the normal subgroup \mathbb{F}_5^2 , and with the natural action of $\text{SL}(2, \mathbb{F}_5)$ on \mathbb{F}_5^2 . (This semidirect product has a presentation with 3 generators and 5 relations, is superperfect, and has order 3000. I do not know whether it is the group of an homology 4-sphere, nor whether it has deficiency 0.)