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## AN HOMOLOGY 4-SPHERE GROUP WITH NEGATIVE DEFICIENCY

by Jonathan A. HILLMAN

ABSTRACT. We give an example to show that homology 4-sphere groups need not have deficiency 0.

The *deficiency*  $\text{def}(G)$  of a finitely presentable group  $G$  is the maximum over all finite presentations  $\mathcal{P}$  for  $G$  of the differences  $g - r$ , where  $g$  is the number of generators and  $r$  is the number of relations in the presentation. It is well-known that  $\text{def}(G)$  may be bounded above by homological invariants [Ep61]. In high dimensions, whether a finitely presentable group can be realized as the fundamental group of an  $n$ -manifold with prescribed homology depends only on the homology of the group; in low dimensions ( $n \leq 4$ ) such conditions remain necessary, while constraints on the deficiency often suffice. However bridging the gap between homologically necessary conditions and combinatorially sufficient conditions is usually a delicate matter. This note considers one such situation.

A group  $G$  is *perfect* if it is equal to its commutator subgroup  $G'$ , i.e., if the abelianization  $G/G' \cong H_1(G; \mathbf{Z})$  is trivial. If  $G$  is the fundamental group of an homology  $n$ -sphere then it is finitely presentable and *superperfect*, i.e.,  $H_1(G; \mathbf{Z}) = H_2(G; \mathbf{Z}) = 0$ . These conditions characterize homology  $n$ -sphere groups for  $n \geq 5$  [Ke69], but in low dimensions more stringent conditions hold. Every perfect group with a presentation of deficiency 0 is an homology 4-sphere group (and therefore is superperfect) [Ke69], but there are finite superperfect groups which are not homology 4-sphere groups [HW85]. As any closed 3-manifold has a handlebody structure with one 0-handle and equal numbers of 1- and 2-handles, homology 3-sphere groups have deficiency 0. However although the finite groups  $\text{SL}(2, \mathbf{F}_p)$  are perfect and have deficiency 0 for each prime  $p \geq 5$  [CR80] the binary icosahedral group  $I^* = \text{SL}(2, \mathbf{F}_5)$  is the only finite homology 3-sphere group.