

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 48 (2002)  
**Heft:** 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** AN HOMOLOGY 4-SPHERE GROUP WITH NEGATIVE DEFICIENCY  
**Autor:** HILLMAN, Jonathan A.  
**Kurzfassung**  
**DOI:** <https://doi.org/10.5169/seals-66076>

#### Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

#### Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

#### Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 06.08.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## AN HOMOLOGY 4-SPHERE GROUP WITH NEGATIVE DEFICIENCY

by Jonathan A. HILLMAN

ABSTRACT. We give an example to show that homology 4-sphere groups need not have deficiency 0.

The *deficiency*  $\text{def}(G)$  of a finitely presentable group  $G$  is the maximum over all finite presentations  $\mathcal{P}$  for  $G$  of the differences  $g - r$ , where  $g$  is the number of generators and  $r$  is the number of relations in the presentation. It is well-known that  $\text{def}(G)$  may be bounded above by homological invariants [Ep61]. In high dimensions, whether a finitely presentable group can be realized as the fundamental group of an  $n$ -manifold with prescribed homology depends only on the homology of the group; in low dimensions ( $n \leq 4$ ) such conditions remain necessary, while constraints on the deficiency often suffice. However bridging the gap between homologically necessary conditions and combinatorially sufficient conditions is usually a delicate matter. This note considers one such situation.

A group  $G$  is *perfect* if it is equal to its commutator subgroup  $G'$ , i.e., if the abelianization  $G/G' \cong H_1(G; \mathbf{Z})$  is trivial. If  $G$  is the fundamental group of an homology  $n$ -sphere then it is finitely presentable and *superperfect*, i.e.,  $H_1(G; \mathbf{Z}) = H_2(G; \mathbf{Z}) = 0$ . These conditions characterize homology  $n$ -sphere groups for  $n \geq 5$  [Ke69], but in low dimensions more stringent conditions hold. Every perfect group with a presentation of deficiency 0 is an homology 4-sphere group (and therefore is superperfect) [Ke69], but there are finite superperfect groups which are not homology 4-sphere groups [HW85]. As any closed 3-manifold has a handlebody structure with one 0-handle and equal numbers of 1- and 2-handles, homology 3-sphere groups have deficiency 0. However although the finite groups  $\text{SL}(2, \mathbf{F}_p)$  are perfect and have deficiency 0 for each prime  $p \geq 5$  [CR80] the binary icosahedral group  $I^* = \text{SL}(2, \mathbf{F}_5)$  is the only finite homology 3-sphere group.