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AN HOMOLOGY 4-SPHERE GROUP WITH NEGATIVE DEFICIENCY

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ABSTRACT. We give an example to show that homology 4-sphere groups need not have deficiency 0.

The *deficiency* def(G) of a finitely presentable group G is the maximum over all finite presentations $\mathcal{P}$ for G of the differences $g-r$, where g is the number of generators and r is the number of relations in the presentation. It is well-known that def(G) may be bounded above by homological invariants [Ep61]. In high dimensions, whether a finitely presentable group can be realized as the fundamental group of an n-manifold with prescribed homology depends only on the homology of the group; in low dimensions ($n \leq 4$) such conditions remain necessary, while constraints on the deficiency often suffice. However bridging the gap between homologically necessary conditions and combinatorially sufficient conditions is usually a delicate matter. This note considers one such situation.

A group G is *perfect* if it is equal to its commutator subgroup G', i.e., if the abelianization $G/G' \cong H_1(G;\mathbf{Z})$ is trivial. If G is the fundamental group of an homology n-sphere then it is finitely presentable and *superperfect*, i.e., $H_1(G;\mathbf{Z}) = H_2(G;\mathbf{Z}) = 0$. These conditions characterize homology n-sphere groups for $n \geq 5$ [Ke69], but in low dimensions more stringent conditions hold. Every perfect group with a presentation of deficiency 0 is an homology 4-sphere group (and therefore is superperfect) [Ke69], but there are finite superperfect groups which are not homology 4-sphere groups [HW85]. As any closed 3-manifold has a handlebody structure with one 0-handle and equal numbers of 1- and 2-handles, homology 3-sphere groups have deficiency 0. However although the finite groups $\mathrm{SL}(2,\mathbf{F}_p)$ are perfect and have deficiency 0 for each prime $p \geq 5$ [CR80] the binary icosahedral group $I^* = \mathrm{SL}(2,\mathbf{F}_5)$ is the only finite homology 3-sphere group.

We shall give an example of an homology 4*-sphere whose group has deficiency* < 0. *Thus none of the implications "G is an homology* 3*-sphere group" $\Rightarrow$ "G is finitely presentable, perfect and* $\mathrm{def}(G) = 0$*" $\Rightarrow$ "G is an homology* 4*-sphere group" $\Rightarrow$ "G is finitely presentable and superperfect" can be reversed.*

A similar outcome was known for knots by the late 1970s. (Namely, none of the implications "G is a 1-knot group" $\Rightarrow$ "G is a high dimensional knot group and $\mathrm{def}(G) = 1$" $\Rightarrow$ "G is a 2-knot group" $\Rightarrow$ "G is a high dimensional knot group" can be reversed [Fo62, Ke65, Fa75]). The issue considered here was raised by Plotnick, who suggested a possible example [Pl82]. (See also [Be02]). We use a related construction, but our example is different, and we do not know whether Plotnick's candidates indeed have negative deficiency.

The construction starts with a 2-knot $K\colon S^2 \to S^4$ and an homology 4-sphere Σ. Let M be the closed 4-manifold obtained by surgery on K, and let $N = M\sharp\Sigma$. Let $G = \pi_1(M)$ and $H = \pi_1(\Sigma)$. (Thus G is the group of the knot K.) Let $t \in G$ represent a generator of $G/G' \cong Z$, and let $h \in H$. The conjugacy class of $th^{-1} \in \pi_1(N) \cong G * H$ is represented by an unique isotopy class of embeddings of S^1 in N. Surgery on such an embedding gives an homology 4-sphere P, with group $\pi = \pi_1(P) = (G*H)/\langle\langle th^{-1}\rangle\rangle$.

Let $\rho = \langle\langle G'\rangle\rangle_\pi$ be the normal closure of the image of G' in π. Then $\pi/\rho \cong H$, and so π is the semidirect product $\rho \rtimes H$. Let $\Gamma = \mathbf{Z}[H]$ and let $I = \mathrm{Ker}(\varepsilon\colon \Gamma \to \mathbf{Z})$ be the augmentation ideal of H. Since H is finitely presentable I has a resolution C_* by free left Γ-modules which are finitely generated in degrees ≤ 2. Let $B = H_1(\pi;\Gamma) \cong \rho/\rho'$. Then B is a left Γ-module and there is an exact sequence $0 \to B \to A \to I \to 0$, in which $A = H_1(\pi, 1;\Gamma)$ is a relative homology group [Cr61]. Evaluating the Jacobian matrix associated to a presentation for π via the natural epimorphism from $\mathbf{Z}[\pi]$ to Γ gives a presentation matrix for A as a module (see [Cr61] or [Fo62]). Thus there is an exact sequence $D_* : \cdots \to \Gamma^m \to \Gamma^n \to A \to 0$, where $n-m = \mathrm{def}(\pi)$. A mapping cone construction leads to an exact sequence of the form $C_2 \oplus D_1 \to C_1 \oplus D_0 \to B \oplus C_0 \to 0$ and hence to a presentation for B of the form $C_2 \oplus D_1 \oplus C_0 \to C_1 \oplus D_0 \to B$.

Now let K be the 2-twist spin of the trefoil knot, with group $G = \langle x, s \mid x^3 = 1,\ sxs^{-1} = x^{-1}\rangle$, and let H be the Higman group with presentation $\langle a,b,c,d \mid bab^{-1} = a^2,\ cbc^{-1} = b^2,\ dcd^{-1} = c^2,\ ada^{-1} = d^2\rangle$. Then H is perfect and $\mathrm{def}(H) = 0$, so there is an homology 4-sphere Σ with group H. Moreover H has cohomological dimension 2 [DV73], and so there is a short exact sequence $0 \to \Gamma^4 \to \Gamma^4 \to I \to 0$. Let $t = s$ and $h = a$. Then $\pi = (G * H)/\langle\langle sa^{-1}\rangle\rangle$ has a presentation of deficiency -1, and $B \cong \Gamma/\Gamma(3, a+1)$. Since $B \cong \Gamma \otimes_\Lambda (\Lambda/\Lambda(3, a+1))$, where $\Lambda = \mathbf{Z}[a, a^{-1}]$, there is an exact sequence

$$0 \to \Gamma \xrightarrow{(3,a+1)} \Gamma^2 \xrightarrow{\begin{pmatrix} a+1 \\ -3 \end{pmatrix}} \Gamma \to B \to 0.$$

Suppose that π has deficiency 0. Then B has deficiency 0 as a left Γ-module, by the general argument above. Hence there is an exact sequence

$$0 \to L \to \Gamma^p \to \Gamma^p \to B \to 0.$$

Schanuel's Lemma gives an isomorphism $\Gamma^{1+p+1} \cong L \oplus \Gamma^{p+2}$, on comparing these two resolutions of B. The endomorphism of Γ^{p+2} given by projection onto the second summand is an automorphism, by a theorem of Kaplansky (see page 122 of [Ka69]). Hence $L = 0$ and so B has a short free resolution. In particular, $\mathrm{Tor}_2^\Gamma(R, B) = 0$ for any right Γ-module R. But it is easily verified that if $\bar{B} \cong \Gamma/(3, a+1)\Gamma$ is the conjugate right Γ-module then $\mathrm{Tor}_2^\Gamma(\bar{B}, B) \neq 0$. Thus our assumption was wrong, and $\mathrm{def}(\pi) = -1 < 0$.

The group of the 2-twist spin of the trefoil knot is the simplest 2-knot group with deficiency 0 [Fo62]. Levine showed that the group of the sum of r copies of this knot has deficiency $1 - r$ [Le78]. If we use this sum in our construction above π now has a presentation of deficiency $-r$ and $B \cong (\Gamma/\Gamma(3, a+1))^r$, so there is an exact sequence

$$0 \to \Gamma^r \to \Gamma^{2r} \to \Gamma^r \to B \to 0.$$

Is $\mathrm{def}(\pi) = -r$?

Is there a *finite* homology 4-sphere group of negative deficiency? Our example above is "very infinite" in the sense that the Higman group H has no finite quotients, and therefore no finite-dimensional representations over any field [Hi51]. The simplest candidate to consider is perhaps the semidirect product of $SL(2, \mathbf{F}_5)$ with the normal subgroup $\mathbf{F}_5^2$, and with the natural action of $SL(2, \mathbf{F}_5)$ on $\mathbf{F}_5^2$. (This semidirect product has a presentation with 3 generators and 5 relations, is superperfect, and has order 3000. I do not know whether it is the group of an homology 4-sphere, nor whether it has deficiency 0.)

REFERENCES

[Be02] BERRICK, A. J. A topologist's view of perfect and acyclic groups. Pages 1–28 in *Invitations to Geometry and Topology* (edited by M. R. Bridson and S. M. Salamon). Oxford University Press, Oxford, 2002.

[CR80] CAMPBELL, C. M. and E. F. ROBERTSON. A deficiency zero presentation for $SL(2,p)$. *Bull. London Math. Soc. 12* (1980), 17–20.

[Cr61] CROWELL, R. H. Corresponding group and module sequences. *Nagoya Math. J. 19* (1961), 27–40.

[DV73] DYER, E. and A. T. VASQUEZ. Some small aspherical spaces. *J. Austral. Math. Soc. 16* (1973), 332–352.

[Ep61] EPSTEIN, D. B. A. Finite presentations of groups and 3-manifolds. *Quart J. Math. Oxford Ser. (2) 12* (1961), 205–212.

[Fa75] FARBER, M. A. Linking coefficients and two-dimensional knots. *Soviet Math. Doklady 16* (1975), 647–650.

[Fo62] FOX, R. H. A quick trip through knot theory. In: *Topology of 3-Manifolds and Related Topics* (edited by M. K. Fort, Jr), 120–167. Prentice-Hall, Englewood Cliffs (N. J.), 1962.

[HW85] HAUSMANN, J.-C. and S. WEINBERGER. Caractéristiques d'Euler et groupes fondamentaux des variétés de dimension 4. *Comment. Math. Helv. 60* (1985), 139–144.

[Hi51] HIGMAN, G. A finitely generated infinite simple group. *J. London Math. Soc. 26* (1951), 61–64.

[Ka69] KAPLANSKY, I. *Fields and Rings.* Chicago University Press, Chicago and London, 1969.

[Ke65] KERVAIRE, M. A. Les nœuds de dimensions supérieures. *Bull. Soc. Math. France 93* (1965), 225–271.

[Ke69] KERVAIRE, M. A. Smooth homology spheres and their fundamental groups. *Trans. Amer. Math. Soc. 144* (1969), 67–72.

[Le78] LEVINE, J. Some results on higher-dimensional knot groups. In: *Knot Theory, Plans-sur-Bex 1977* (edited by J.-C. Hausmann), Lecture Notes in Math. *685*, 243–269. Springer Verlag, 1978.

[Pl82] PLOTNICK, S. Circle actions and fundamental groups for homology 4-spheres. *Trans. Amer. Math. Soc. 273* (1982), 393–404.

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