

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 48 (2002)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: TOPOLOGICAL PROOF OF THE GROTHENDIECK FORMULA IN REAL ALGEBRAIC GEOMETRY
Autor: BOCHNAK, J. / Kucharz, W.
Kapitel: 2. Proof of the Grothendieck formula
DOI: <https://doi.org/10.5169/seals-66075>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 07.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

classes in codimension 2 by Zariski closed subsets. More precisely, we have the following result.

THEOREM 1.6. *Let M be a compact orientable smooth manifold of dimension at least 5 and let G be a subgroup of $H^2(M; \mathbf{Z}/2)$. Then the following conditions are equivalent:*

(a) *There exist a nonsingular real algebraic variety X and a diffeomorphism $\varphi: X \rightarrow M$ such that $\varphi^*(G) = H_{\text{alg}}^2(X; \mathbf{Z}/2)$.*

(b) *$w_2(M) \in G \subseteq W^2(M)$, where $w_2(M)$ is the second Stiefel-Whitney class of M .*

Proof. See [13].

Another application concerns the problem of approximation of smooth curves (that is, one-dimensional smooth submanifolds) by algebraic curves. First recall that a compact smooth submanifold N of a nonsingular real algebraic variety X is said to *admit an algebraic approximation* in X if for each neighborhood \mathcal{U} of the inclusion map $N \hookrightarrow X$ (in the \mathcal{C}^∞ topology on the set $\mathcal{C}^\infty(N, X)$ of smooth maps from N into X), there exists a smooth embedding $e: N \rightarrow X$ such that e is in \mathcal{U} and $e(N)$ is a nonsingular Zariski closed subset of X .

THEOREM 1.7. *Let X be a compact nonsingular real algebraic variety of dimension 3 and let C be a compact smooth curve in X . Then C admits an algebraic approximation in X if and only if the $\mathbf{Z}/2$ -homology class represented by C is in $H_1^{\text{alg}}(X; \mathbf{Z}/2)$.*

The proof of Theorem 1.7 will be given elsewhere. Under the extra assumption that C is connected and homologous to the union of finitely many nonsingular real algebraic curves in X the theorem is proved in [4].

2. PROOF OF THE GROTHENDIECK FORMULA

We shall use homology and cohomology groups with coefficients exclusively in $\mathbf{Z}/2$ and therefore we shall suppress the coefficient group in our notation.

For any continuous map $f: (X, A) \rightarrow (Y, B)$ between pairs of topological spaces, we let

$$f_*: H_k(X, A) \rightarrow H_k(Y, B), \quad f^*: H^k(Y, B) \rightarrow H^k(X, A)$$

denote the induced homomorphisms.

For the convenience of the reader we shall now review some facts from topology. Let B be a paracompact topological space and let $\xi = (E, \pi, B)$ be a real vector bundle of rank k on B . Let $s_0: B \rightarrow E$ be the zero section of ξ , that is, $s_0(x) = 0_x$ for all x in B , where 0_x is the zero vector in the fiber $E_x = \pi^{-1}(x)$. We set $0_E = s_0(B)$. Recall that the Thom class τ_ξ of ξ is a unique element of $H^k(E, E \setminus 0_E)$ such that for every point x in B , the homomorphism

$$H^k(E, E \setminus 0_E) \rightarrow H^k(E_x, E_x \setminus \{0_x\}) \cong \mathbf{Z}/2,$$

induced by the inclusion map $(E_x, E_x \setminus \{0_x\}) \hookrightarrow (E, E \setminus 0_E)$, sends τ_ξ to the generator of $\mathbf{Z}/2$ [24, Theorem 8.1] (the name "Thom class" is not used in [24]). For every nonnegative integer q , we have the Thom isomorphism

$$\begin{aligned} \varphi_q: H^q(B) &\rightarrow H^{k+q}(E, E \setminus 0_E) \\ \varphi_q(v) &= \pi^*(v) \cup \tau_\xi \quad \text{for all } v \text{ in } H^q(B) \end{aligned}$$

[24, Definition 8.2].

If $s: B \rightarrow E$ is any continuous section of ξ and $\bar{s}: (B, B \setminus s^{-1}(0_E)) \rightarrow (E, E \setminus 0_E)$ is the map defined by s , then

$$(2.1) \quad w_k(\xi) = i^*(\bar{s}^*(\tau_\xi)),$$

where $i: B = (B, \emptyset) \hookrightarrow (B, B \setminus s^{-1}(0_E))$ is the inclusion map. Indeed, let $j: E \hookrightarrow (E, E \setminus 0_E)$ be the inclusion map. Note that $H: E \times [0, 1] \rightarrow (E, E \setminus 0_E)$, defined by $H(e, t) = (1-t)j(e) + t(\bar{s} \circ i \circ \pi)(e)$ for all (e, t) in $E \times [0, 1]$, is a homotopy between j and $\bar{s} \circ i \circ \pi$. In particular, $j^* = (\bar{s} \circ i \circ \pi)^* = \pi^* \circ i^* \circ \bar{s}^*$, and hence

$$\pi^*(i^*(\bar{s}^*(\tau_\xi))) \cup \tau_\xi = j^*(\tau_\xi) \cup \tau_\xi = \tau_\xi \cup \tau_\xi,$$

where the last equality is the standard property of the cup product [26, p. 251, property 8]. Thus $\varphi_k(i^*(\bar{s}^*(\tau_\xi))) = \tau_\xi \cup \tau_\xi$. Now, (2.1) follows since $w_k(\xi) = \varphi_k^{-1}(\tau_\xi \cup \tau_\xi)$ [24, p. 91].

Let M be a smooth m -dimensional manifold and let N be a smooth n -dimensional submanifold of M . Assume that N is a closed subset of M . A tubular neighborhood of N in M is a smooth real vector bundle $\xi = (E, \pi, N)$ on N such that E is an open neighborhood of N in M and $0_E = N$ [20]. By

the excision property, the inclusion map $e: (E, E \setminus N) \hookrightarrow (M, M \setminus N)$ induces an isomorphism

$$e^*: H^k(M, M \setminus N) \rightarrow H^k(E, E \setminus N),$$

where $k = m - n$. The Thom class τ_N^M of N in M is a unique element of $H^k(M, M \setminus N)$ such that $e^*(\tau_N^M) = \tau_\xi$. The Thom isomorphism yields

$$H^k(M, M \setminus N) \cong H^0(N).$$

Hence

$$(2.2) \quad \tau_N^M \text{ generates } H^k(M, M \setminus N) \cong \mathbf{Z}/2,$$

provided N is connected. Assuming that N has exactly r connected components N_1, \dots, N_r , the inclusion maps $e_i: (M, M \setminus N) \hookrightarrow (M, M \setminus N_i)$ give rise to an isomorphism

$$t: \bigoplus_{i=1}^r H^k(M, M \setminus N_i) \rightarrow H^k(M, M \setminus N)$$

$$t(u_1, \dots, u_r) = e_1^*(u_1) + \dots + e_r^*(u_r)$$

satisfying

$$(2.3) \quad t(\tau_{N_1}^M, \dots, \tau_{N_r}^M) = \tau_N^M.$$

If $f: M \rightarrow P$ is a smooth map between smooth manifolds, transverse to a smooth submanifold Q of P (Q a closed subset of P) and with $N = f^{-1}(Q)$, then

$$(2.4) \quad \bar{f}^*(\tau_Q^P) = \tau_N^M,$$

where $\bar{f}: (M, M \setminus N) \rightarrow (P, P \setminus Q)$ is the map defined by f . Indeed, after a homotopy, f looks like a vector bundle map between tubular neighborhoods of N and Q [20, p.117, Theorem 6.7], and hence (2.4) follows from the definition of the Thom class.

Let Δ be the diagonal of $M \times M$,

$$\Delta = \{(x, y) \in M \times M \mid x = y\},$$

and let τ in $H^m(M \times M, (M \times M) \setminus \Delta)$ be the Thom class of Δ in $M \times M$. For every point x in M , the image of τ under the homomorphism

$$H^m(M \times M, (M \times M) \setminus \Delta) \rightarrow H^m(M, M \setminus \{x\}) \cong \mathbf{Z}/2$$

induced by the map $(M, M \setminus \{x\}) \rightarrow (M \times M, (M \times M) \setminus \Delta)$, $y \rightarrow (x, y)$, generates $\mathbf{Z}/2$ [24, Lemma 11.7]. Thus τ is the orientation class of M over $\mathbf{Z}/2$ in

the terminology used in [26, p.294]. For any pair (A, B) of subsets of M , $B \subseteq A$, and any integer q satisfying $0 \leq q \leq m$, let

$$\gamma_{A,B}: H_q(A, B) \rightarrow H^{m-q}(M \setminus B, M \setminus A)$$

be the homomorphism defined by

$$\gamma_{A,B}(a) = a \setminus j_{A,B}^*(\tau),$$

where \setminus is the slant product and

$$j_{A,B}: (A \times (M \setminus B), (A \times (M \setminus A)) \cup (B \times (M \setminus B))) \hookrightarrow (M \times M, (M \times M) \setminus \Delta)$$

is the inclusion map, cf. [26, p.351]. If B is empty, we shall write γ_A instead of $\gamma_{A,\emptyset}$. The following naturality property is satisfied: if (A', B') is another pair of subsets of M , $B' \subseteq A'$, and $A \subseteq A'$, $B \subseteq B'$, then the diagram

$$(2.5) \quad \begin{array}{ccc} H_q(A, B) & \xrightarrow{\gamma_{A,B}} & H^{m-q}(M \setminus B, M \setminus A) \\ \downarrow & & \downarrow \\ H_q(A', B') & \xrightarrow{\gamma_{A',B'}} & H^{m-q}(M \setminus B', M \setminus A'), \end{array}$$

where the vertical homomorphisms are induced by the appropriate inclusion maps, is commutative [26, pp.287, 289, 351]. Furthermore, if M is compact, then

$$(2.6) \quad \gamma_M = D_M^{-1},$$

that is,

$$\gamma_M: H_q(M) \rightarrow H^{m-q}(M)$$

is the inverse of the Poincaré duality isomorphism

$$D_M: H^{m-q}(M) \rightarrow H_q(M), \quad D_M(u) = u \cap [M].$$

This follows from [26, p.305, Theorem 12] and the fact that, in the notation of [26, p.353, Lemma 15], θ is the identity map, provided $X = Y$, $G = \mathbb{Z}/2$.

We shall also make use of the following result.

PROPOSITION 2.7. *If M is compact and (A, B) is a compact polyhedral pair in M , then*

$$\gamma_{A,B}: H_q(A, B) \rightarrow H^{m-q}(M \setminus B, M \setminus A)$$

is an isomorphism.

Proof. We have the following diagram:

$$\begin{array}{ccc}
 H_q(B) & \xrightarrow{\gamma_B} & H^{m-q}(M, M \setminus B) \\
 \downarrow & & \downarrow \\
 H_q(A) & \xrightarrow{\gamma_A} & H^{m-q}(M, M \setminus A) \\
 \downarrow & & \downarrow \\
 H_q(A, B) & \xrightarrow{\gamma_{A,B}} & H^{m-q}(M \setminus B, M \setminus A) \\
 \downarrow & & \downarrow \\
 H_{q-1}(B) & \xrightarrow{\gamma_B} & H^{m-q+1}(M, M \setminus B) \\
 \downarrow & & \downarrow \\
 H_{q-1}(A) & \xrightarrow{\gamma_A} & H^{m-q+1}(M, M \setminus A),
 \end{array}$$

where the columns are parts of the long exact sequences for the pair (A, B) and the triple $(M, M \setminus B, M \setminus A)$. By (2.5) and [26, p. 287, property 3, and p. 351], the diagram is commutative. It is proved in [26, p. 351, Lemma 14] that γ_A and γ_B are isomorphisms for q and $q-1$. In view of the five lemma, $\gamma_{A,B}$ is also an isomorphism. \square

After this preparation, we are ready to prove an auxiliary result relating homology and cohomology of real algebraic varieties. Let X be a compact n -dimensional nonsingular real algebraic variety and let V be a d -dimensional Zariski closed subset of X . By Theorem 1.1, V is a compact polyhedron and hence

$$\gamma_V: H_d(V) \rightarrow H^c(X, X \setminus V),$$

where $c = n - d$, is an isomorphism in view of Proposition 2.7. For our purposes it is important to give a characterization of $\gamma_V([V])$. Set $S = \text{Sing}(V)$ and let

$$i: (X \setminus S, (X \setminus S) \setminus (V \setminus S)) \hookrightarrow (X, X \setminus V), \quad j: X \hookrightarrow (X, X \setminus V)$$

be the inclusion maps (of course, $X \setminus V = (X \setminus S) \setminus (V \setminus S)$). Since $V \setminus S$ is a d -dimensional nonsingular Zariski closed subset of $X \setminus S$, the Thom class $\tau_{V \setminus S}^{X \setminus S}$ in $H^c(X \setminus S, (X \setminus S) \setminus (V \setminus S))$ is defined.

PROPOSITION 2.8. *There exists a unique element τ_V^X in $H^c(X, X \setminus V)$ such that*

$$i^*(\tau_V^X) = \tau_{V \setminus S}^{X \setminus S}.$$

Furthermore,

$$\tau_V^X = \gamma_V([V]) \quad \text{and} \quad D_X(j^*(\tau_V^X)) = [V]_X.$$

Proof. We shall first prove $i^*(\gamma_V([V])) = \tau_{V \setminus S}^{X \setminus S}$. The smooth manifold $V \setminus S$ is a semialgebraic set and therefore has finitely many connected components, say N_1, \dots, N_r [11, p.35]. If V_i is the closure of N_i in V and $S_i = V_i \cap S$, then $N_i = V_i \setminus S_i$. Note that V_i and S_i are compact semialgebraic subsets of V [8, p.61 or 11, p.27]. By (2.5), we have the following commutative diagram:

$$\begin{array}{ccccc} H_d(V) & \xrightarrow{\varphi} & H_d(V, S) & \xleftarrow{\alpha} & \bigoplus_{i=1}^r H_d(V_i, S_i) \\ \gamma_V \downarrow & & \gamma_{V, S} \downarrow & & \bigoplus_{i=1}^r \gamma_{V_i, S_i} \downarrow \\ H^c(X, X \setminus V) & \xrightarrow{i^*} & H^c(X \setminus S, (X \setminus S) \setminus (V \setminus S)) & \xleftarrow{\beta} & \bigoplus_{i=1}^r H^c(X \setminus S_i, (X \setminus S_i) \setminus N_i), \end{array}$$

where φ is induced by the appropriate inclusion map, whereas

$$\alpha(a_1, \dots, a_r) = \alpha_1(a_1) + \dots + \alpha_r(a_r),$$

$$\beta(u_1, \dots, u_r) = \beta_1(u_1) + \dots + \beta_r(u_r),$$

with

$$\alpha_i: H_d(V_i, S_i) \rightarrow H_d(V, S)$$

$$\beta_i: H^c(X \setminus S_i, (X \setminus S_i) \setminus N_i) \rightarrow H^c(X \setminus S, (X \setminus S) \setminus (V \setminus S))$$

induced by the inclusion maps.

Since N_1, \dots, N_r are the connected components of the smooth manifold $V \setminus S$, we have another commutative diagram:

$$\begin{array}{ccc} H^c(X \setminus S, (X \setminus S) \setminus (V \setminus S)) & \xleftarrow{\beta} & \bigoplus_{i=1}^r H^c(X \setminus S_i, (X \setminus S_i) \setminus N_i) \\ \uparrow t & & \downarrow \bigoplus_{i=1}^r \psi_i \\ \bigoplus_{i=1}^r H^c(X \setminus S, (X \setminus S) \setminus N_i) & \xleftarrow{id} & \bigoplus_{i=1}^r H^c(X \setminus S, (X \setminus S) \setminus N_i), \end{array}$$

where

$$\psi_i: H^c(X \setminus S_i, (X \setminus S_i) \setminus N_i) \rightarrow H^c(X \setminus S, (X \setminus S) \setminus N_i)$$

is the homomorphism induced by the appropriate inclusion map and t is the isomorphism of (2.3). It follows from the definition of the Thom class that

$$(a) \quad \psi_i(\tau_{N_i}^{X \setminus S_i}) = \tau_{N_i}^{X \setminus S}.$$

Hence, in view of (2.2), ψ_i is an isomorphism of cyclic groups isomorphic to $\mathbf{Z}/2$. Applying (2.3) and (a), we get

$$(b) \quad \beta(\tau_{N_1}^{X \setminus S_1}, \dots, \tau_{N_r}^{X \setminus S_r}) = \tau_{V \setminus S}^{X \setminus S}.$$

Since, by Proposition 2.7, γ_{V_i, S_i} is an isomorphism, the group $H_d(V_i, S_i)$ is isomorphic to $\mathbf{Z}/2$; let a_i be its unique generator. Now, (a) and (b) imply

$$\gamma_{V, S}(\alpha(a_1, \dots, a_r)) = \tau_{V \setminus S}^{X \setminus S}.$$

Thus in order to verify $i^*(\gamma_V([V])) = \tau_{V \setminus S}^{X \setminus S}$ it suffices to prove

$$(c) \quad \alpha(a_1, \dots, a_r) = \varphi([V]),$$

which can be done as follows.

Let $\Phi: |K| \rightarrow V$ be a semialgebraic triangulation of V compatible with $\{V_1, \dots, V_r, S_1, \dots, S_r\}$ (Theorem 1.1). Denote by c_i the chain which is the sum of all d -simplices of K whose images under Φ are contained in V_i . Since $N_i = V_i \setminus S_i$ is a smooth d -dimensional manifold, it follows that every open $(d-1)$ -simplex σ of K with $\Phi(\sigma)$ contained in N_i is a face of exactly two d -simplices of K . Thus c_i represents a nonzero homology class in $H_d(V_i, S_i) \cong \mathbf{Z}/2$; in other words, c_i represents a_i . On the other hand, $c_1 + \dots + c_r$ is the sum of all d -simplices of K and therefore it is a cycle representing the fundamental class $[V]$ in $H_d(V)$. Hence (c) follows and $i^*(\gamma_V([V])) = \tau_{V \setminus S}^{X \setminus S}$ is proved.

Let us observe that i^* is injective. Indeed, there is an exact sequence

$$\dots \rightarrow H^c(X, X \setminus S) \rightarrow H^c(X, X \setminus V) \rightarrow H^c(X \setminus S, X \setminus V) \rightarrow \dots$$

corresponding to the triple $(X, X \setminus S, X \setminus V)$. By Proposition 2.7, $\gamma_S: H_d(S) \rightarrow H^c(X, X \setminus S)$ is an isomorphism. Since $\dim S < d$, we obtain $H_d(S) = 0$, which implies $H^c(X, X \setminus S) = 0$. Hence i^* is injective as asserted.

Thus $\tau_V^X = \gamma_V([V])$ is a unique element of $H^c(X, X \setminus V)$ satisfying $i^*(\tau_V^X) = \tau_{V \setminus S}^{X \setminus S}$.

It remains to prove $D_X(j^*(\tau_V^X)) = [V]_X$. By (2.5), we have the following commutative diagram:

$$\begin{array}{ccc} H_d(V) & \xrightarrow{e_*} & H_d(X) \\ \gamma_V \downarrow & & \downarrow \gamma_X \\ H^c(X, X \setminus V) & \xrightarrow{j^*} & H^c(X), \end{array}$$

where $e: V \hookrightarrow X$ is the inclusion map. In view of (2.6), γ_X is the inverse of D_X and we obtain $D_X(j^*(\tau_V^X)) = e_*([V]) = [V]_X$. Thus the proof is complete. \square

We shall now recall a purely algebraic result. Definitions of algebraic terms not explained here can all be found in [23]. Given a ring R (commutative with identity), we let $K_0(R)$ denote the Grothendieck group of finitely generated projective R -modules. If S is a multiplicatively closed subset of R and $S^{-1}R$ denotes the ring of fractions of R with denominators in S , then the canonical ring homomorphism $j_S: R \rightarrow S^{-1}R$, $j_S(r) = r/1$, induces a group homomorphism $K_0(R) \rightarrow K_0(S^{-1}R)$. Assuming that R is a regular ring of finite Krull dimension, every finitely generated R -module has a finite projective resolution [23, p. 208]. The last fact allows one to apply [6, p. 453, Proposition 2.1, p. 492, Proposition 6.1], which yields the result we require: the homomorphism $K_0(R) \rightarrow K_0(S^{-1}R)$ is surjective, provided that R is a regular ring of finite Krull dimension (this also easily follows from [23, p. 210, Exercise 4]).

To make use of this result we need some algebraic properties of the ring $\mathcal{R}(X)$ of regular functions on a real algebraic variety X . Suppose that X is a Zariski locally closed subset of \mathbf{R}^n and let $\mathcal{P}(X)$ be the ring of polynomial functions from X into \mathbf{R} ($f: X \rightarrow \mathbf{R}$ is a polynomial function if for some polynomial P in $\mathbf{R}[T_1, \dots, T_n]$, one has $f(x) = P(x)$ for all x in X). Clearly, $\mathcal{P}(X)$ is a finitely generated \mathbf{R} -algebra and thus a Noetherian ring [23, p. 11]. Furthermore, the Krull dimension of $\mathcal{P}(X)$ is equal to $\dim X$ [11, p. 50]. Recall that $\mathcal{R}(X)$ consists of all functions of the form f/g , where f, g are in $\mathcal{P}(X)$ and $g^{-1}(0) = \emptyset$. In other words, $\mathcal{R}(X)$ is the ring of fractions of $\mathcal{P}(X)$ with denominators in the set $\{g \in \mathcal{P}(X) \mid g^{-1}(0) = \emptyset\}$. It follows that $\mathcal{R}(X)$ is a Noetherian ring of Krull dimension $\dim X$ [23, p. 81]. Obviously, for every point x in X ,

$$m_x = \{f \in \mathcal{R}(X) \mid f(x) = 0\}$$

is a maximal ideal of $\mathcal{R}(X)$ and each maximal ideal of $\mathcal{R}(X)$ is equal to m_x for some x . The localization $\mathcal{R}(X)_x$ of $\mathcal{R}(X)$ with respect to m_x is a Noetherian local ring of Krull dimension not exceeding $\dim X$ [23, p. 81]. A point x in X is nonsingular if and only if the local ring $\mathcal{R}(X)_x$ is regular of Krull dimension $\dim X$ [11, p. 67]. In particular, the ring $\mathcal{R}(X)$ is regular of finite Krull dimension, provided X is nonsingular. Given a Zariski open subset U of X , the subset

$$S(U) = \{g \in \mathcal{R}(X) \mid g^{-1}(0) \subseteq X \setminus U\}$$

of $\mathcal{R}(X)$ is multiplicatively closed. Since $\mathcal{R}(U) = S(U)^{-1}\mathcal{R}(X)$, it follows from the facts reviewed above that the group homomorphism

$$(2.9) \quad K_0(\mathcal{R}(X)) \rightarrow K_0(\mathcal{R}(U)),$$

induced by the restriction ring homomorphism $\mathcal{R}(X) \rightarrow \mathcal{R}(U)$, $f \rightarrow f|_U$, is surjective, assuming X is nonsingular.

PROPOSITION 2.10. *Let X be a nonsingular real algebraic variety and let U be a Zariski open subset of X . For any algebraic vector bundle η on U , there exists an algebraic vector bundle ξ on X such that $\xi|_U$ and η are algebraically stably equivalent (that is, one can find algebraically trivial vector bundles ϵ_1 and ϵ_2 on U with the property that the bundles $(\xi|_U) \oplus \epsilon_1$ and $\eta \oplus \epsilon_2$ on U are algebraically isomorphic).*

Proof. Let Y be a real algebraic variety. For any algebraic vector bundle ζ on Y , let $\Gamma(\zeta)$ denote the $\mathcal{R}(Y)$ -module of algebraic global sections of ζ . One readily proves that the correspondence $\zeta \rightarrow \Gamma(\zeta)$ establishes an equivalence of the category of algebraic vector bundles on Y with the category of finitely generated projective $\mathcal{R}(Y)$ -modules [11, Proposition 12.1.12]. The proposition follows since (2.9) is surjective. \square

Let Y be a real algebraic variety and let W be a Zariski closed subset of Y . Denote by $I_Y(W)$ the ideal of $\mathcal{R}(Y)$ consisting of all regular functions vanishing on W ,

$$I_Y(W) = \{f \in \mathcal{R}(Y) \mid f(y) = 0 \text{ for all } y \text{ in } W\}.$$

The restriction homomorphism $\mathcal{R}(Y) \rightarrow \mathcal{R}(W)$, $f \rightarrow f|_W$, gives rise, for each point y in W , to a ring epimorphism $\mathcal{R}(Y)_y \rightarrow \mathcal{R}(W)_y$, whose kernel is equal to the ideal $I_Y(W)\mathcal{R}(Y)_y$ of $\mathcal{R}(Y)_y$. In particular, the quotient ring $\mathcal{R}(Y)_y/I_Y(W)\mathcal{R}(Y)_y$ is isomorphic to $\mathcal{R}(W)_y$. Therefore if y in W is a nonsingular point of Y and $k = \dim Y - \dim W$, then given elements f_1, \dots, f_k of $I_Y(W)$, the following conditions are equivalent:

- (i) $I_Y(W)\mathcal{R}(Y)_y = (f_1, \dots, f_k)\mathcal{R}(Y)_y$ and y is a nonsingular point of W ,
- (ii) $I_Y(W)\mathcal{R}(Y)_y = (f_1, \dots, f_k)\mathcal{R}(Y)_y$ and there exist elements f_{k+1}, \dots, f_{k+d} of $\mathcal{R}(Y)$, $d = \dim W$, such that f_1, \dots, f_{k+d} generate the unique maximal ideal of the local ring $\mathcal{R}(Y)_y$,
- (iii) the map $(f_1, \dots, f_k): Y \setminus \text{Sing}(Y) \rightarrow \mathbf{R}^k$ is transverse to 0 at y and $W \cap H = f_1^{-1}(0) \cap \dots \cap f_k^{-1}(0) \cap H$, where H is a Zariski open neighborhood of y in $Y \setminus \text{Sing}(Y)$.

Indeed, the equivalence of (i) and (ii) is a consequence of [23, p. 169, Proposition 1.10]. Furthermore, f_1, \dots, f_{k+d} generate the maximal ideal of $\mathcal{R}(Y)_y$ if and only if there exists a neighborhood N of y in $Y \setminus \text{Sing}(Y)$ such

that the restriction of (f_1, \dots, f_{k+d}) to N is a local coordinate system for the smooth manifold $Y \setminus \text{Sing}(Y)$ [11, pp. 66, 67]. Hence the equivalence of (ii) and (iii) easily follows.

It also follows from [23, p. 169, Proposition 1.10] that $I_Y(W)\mathcal{R}(Y)_y$ is generated by k elements, provided y in W is a nonsingular point of Y and of W .

We shall freely use the facts just reviewed.

Proof of Theorem 1.5. By assumption, $D_X(v) = [V]_X$, where V is a Zariski closed subset of X with $\dim X - \dim V = 2$. If V_1, \dots, V_p are the irreducible components of V of dimension $\dim V$, then $[V]_X = [V_1]_X + \dots + [V_p]_X$, and hence it suffices to prove the theorem assuming that V is irreducible.

Let x_0 be a nonsingular point of V . Then the ideal $I_X(V)\mathcal{R}(X)_{x_0}$ of the ring $\mathcal{R}(X)_{x_0}$ can be generated by two elements; we choose generators a_1, a_2 that belong to $I_X(V)$. Hence there exists a Zariski open neighborhood U of x_0 in X such that the ideal $I_X(V)\mathcal{R}(U)$ of the ring $\mathcal{R}(U)$ is generated by a_1 and a_2 . This implies

$$(a) \quad I_X(V)\mathcal{R}(U)_x = (a_1, a_2)\mathcal{R}(U)_x \text{ for all } x \text{ in } U.$$

Since $\text{Sing}(V)$ is Zariski closed in V , shrinking U if necessary, we may assume that $U \cap \text{Sing}(V) = \emptyset$. Hence from (a), we obtain

$$(b) \quad \text{the map } (a_1, a_2): U \rightarrow \mathbf{R}^2 \text{ is transverse to } 0 \text{ in } \mathbf{R}^2 \\ \text{at each point } x \text{ in } U \cap V.$$

Setting $S = V \setminus (U \cap V)$, we have $\text{Sing}(V) \subseteq S$ and, by virtue of irreducibility of V ,

$$(c) \quad \dim S < \dim V.$$

Let $Y = X \setminus S$ and $W = V \setminus S$. Then Y is a Zariski open subset of X and W is a Zariski closed subset of Y , with $\dim Y - \dim W = 2$.

CLAIM. *There exist an algebraic vector bundle $\eta = (E, \pi, Y)$ on Y and an algebraic section $s: Y \rightarrow E$ of η such that η is of rank 2, $W = s^{-1}(0_E)$, and s is transverse to 0_E .*

We prove the claim as follows. Choose a regular function b in $\mathcal{R}(Y)$ with $b^{-1}(0) = W$. Set $b_k = a_k|_Y$ for $k = 1, 2$, and define a map $F: Y \times \mathbf{R}^2 \rightarrow \mathbf{R}^2$

by

$$F(y, t) = F_t(y) = (b_1(y) + t_1 b(y)^2, b_2(y) + t_2 b(y)^2)$$

for all y in Y and $t = (t_1, t_2)$ in \mathbf{R}^2 .

We assert that F is transverse to 0 in \mathbf{R}^2 . Indeed, suppose $F(y, t) = 0$ for some (y, t) in $Y \times \mathbf{R}^2$. If y is not in W , then the assertion holds since it suffices to consider the partial derivatives with respect to t_1 and t_2 . If y is in W , then (b) implies that $F_t: Y \rightarrow \mathbf{R}^2$ is transverse to 0 in \mathbf{R}^2 at y , which means that the assertion also holds in this case. Hence the assertion is proved.

It follows from the assertion and a standard transversality theorem [20, p. 79, Theorem 2.7] that there exists a point t in \mathbf{R}^2 for which the map

$$F_t = (f_1, f_2): Y \rightarrow \mathbf{R}^2$$

is transverse to 0 in \mathbf{R}^2 . Since f_1 and f_2 are in $I_Y(W)$ and W is nonsingular, we get

$$I_Y(W)\mathcal{R}(Y)_y = (f_1, f_2)\mathcal{R}(Y)_y$$

for all y in W . Hence for each point y in W , one can find a Zariski open neighborhood G_y of y in Y with

$$I_Y(W)\mathcal{R}(G_y) = (f_1, f_2)\mathcal{R}(G_y).$$

In particular, $W \cap G_y = f_1^{-1}(0) \cap f_2^{-1}(0) \cap G_y$. Taking G to be the union of the G_y for y in W , we get $W = f_1^{-1}(0) \cap f_2^{-1}(0) \cap G$, which implies

$$(d) \quad f_1^{-1}(0) \cap f_2^{-1}(0) = W \cup W',$$

where W' is a subset of Y disjoint from W . Clearly, W' is contained in $Y \setminus G$. Since $W \cup W'$ and $Y \setminus G$ are Zariski closed subsets of Y , and $W' = (W \cup W') \cap (Y \setminus G)$, it follows that W' is also Zariski closed in Y . The transversality of $(f_1, f_2): Y \rightarrow \mathbf{R}^2$ to 0 in \mathbf{R}^2 together with (d) imply

$$(e) \quad I_Y(W \cup W')\mathcal{R}(Y)_y = (f_1, f_2)\mathcal{R}(Y)_y \text{ for all } y \text{ in } Y.$$

Choosing regular functions ψ_1 and ψ_2 in $\mathcal{R}(Y)$ with $\psi_1^{-1}(0) = W$ and $\psi_2^{-1}(0) = W'$ (this is possible since W and W' are Zariski closed in Y), we see that $\psi_1\psi_2$ belongs to $I_Y(W \cup W')$ and hence

$$\psi_1\psi_2 = h_1f_1 + h_2f_2$$

for some regular functions h_1 and h_2 in $\mathcal{R}(Y)$ (the last assertion can easily be deduced directly from (e), but, anyhow, it is also a consequence of (e) and [23, p. 93, Rule 1.1]).

Let $\mathbf{M}_2(\mathbf{R})$ denote the set of all real 2×2 matrices (identified with \mathbf{R}^4 and regarded as a real algebraic variety). Consider regular maps $g_{21}: U_1 = Y \setminus W \rightarrow \mathbf{M}_2(\mathbf{R})$ and $g_{12}: U_2 = Y \setminus W' \rightarrow \mathbf{M}_2(\mathbf{R})$ defined by

$$g_{21} = \begin{bmatrix} f_1\psi_2/\psi_1 & -h_2/\psi_1^2 \\ f_2\psi_2/\psi_1 & h_1/\psi_1^2 \end{bmatrix}, \quad g_{12} = \begin{bmatrix} h_1/\psi_2^2 & h_2/\psi_2^2 \\ -f_2\psi_1/\psi_2 & f_1\psi_1/\psi_2 \end{bmatrix}.$$

For each point y in $U_1 \cap U_2$, the matrices $g_{12}(y)$ and $g_{21}(y)$ are invertible and $g_{12}(y)g_{21}(y)$ is the identity matrix. Define

$$E = \{(y, v_1, v_2) \in Y \times \mathbf{R}^2 \times \mathbf{R}^2 \mid v_1 = g_{12}(y)v_2 \text{ if } y \in U_2 \\ \text{and } v_2 = g_{21}(y)v_1 \text{ if } y \in U_1\}$$

and $\pi: E \rightarrow Y$, $\pi(y, v_1, v_2) = y$. Since $\{U_1, U_2\}$ is a Zariski open cover of Y , it follows that E is a Zariski closed subset of $Y \times \mathbf{R}^2 \times \mathbf{R}^2$. Clearly, π is a regular map and, for each point y in Y , the fiber $E_y = \pi^{-1}(y)$ is a vector subspace of $\{y\} \times \mathbf{R}^2 \times \mathbf{R}^2$. Furthermore, the map

$$U_k \times \mathbf{R}^2 \rightarrow \pi^{-1}(U_k), (y, v) \rightarrow (y, g_{1k}(y) \cdot v, g_{2k}(y) \cdot v)$$

is biregular for $k = 1, 2$, where $g_{kk}(y)$ is the identity matrix. Thus $\eta = (E, \pi, Y)$ is an algebraic vector bundle of rank 2 on Y . The map $s: Y \rightarrow E$

$$s(y) = (y, (\psi_1(y), 0), (f_1(y)\psi_2(y), f_2(y)\psi_2(y)))$$

is an algebraic section of η with $s^{-1}(0_E) = W$. On U_2 the section s is represented by $(f_1, f_2): U_2 \rightarrow \mathbf{R}^2$, and therefore s is transverse to 0_E . Hence the claim is proved.

Let $\bar{s}: (Y, Y \setminus W) \rightarrow (E, E \setminus 0_E)$ be the map defined by s and let $\ell: Y \hookrightarrow (Y, Y \setminus W)$ be the inclusion map. In view of (2.1), we have $w_2(\eta) = \ell^*(\bar{s}^*(\tau_\eta))$, while (2.4) yields $\bar{s}^*(\tau_\eta) = \tau_W^Y$. It follows that

$$(f) \quad w_2(\eta) = \ell^*(\tau_W^Y).$$

If $i: (Y, Y \setminus W) \hookrightarrow (X, X \setminus V)$, $j: X \hookrightarrow (X, X \setminus V)$, and $e: Y \hookrightarrow X$ are the inclusion maps, then the diagram

$$\begin{array}{ccc} H^2(X, X \setminus V) & \xrightarrow{i^*} & H^2(Y, Y \setminus W) \\ j^* \downarrow & & \downarrow \ell^* \\ H^2(X) & \xrightarrow{e^*} & H^2(Y) \end{array}$$

is commutative.

Since $W \subseteq V \setminus \text{Sing}(V)$, Proposition 2.8 yields

$$(g) \quad i^*(\tau_V^X) = \tau_W^Y, \quad j^*(\tau_V^X) = v.$$

By combining (d) and (e), we get

$$(h) \quad w_2(\eta) = \ell^*(i^*(\tau_V^X)) = e^*(j^*(\tau_V^X)) = e^*(v).$$

Proposition 2.10 implies that there exists an algebraic vector bundle ζ on X , whose restriction to Y is algebraically stably equivalent to η . In particular, $w_2(\eta) = w_2(\zeta|Y) = e^*(w_2(\zeta))$, and hence applying (h), we get

$$(i) \quad e^*(v) = e^*(w_2(\zeta)).$$

Note that e^* is injective. Indeed, there is an exact sequence

$$H^2(X, Y) \longrightarrow H^2(X) \xrightarrow{e^*} H^2(Y).$$

Since $S = X \setminus Y$ is Zariski closed in X , by Theorem 1.1 and Proposition 2.7, $H^2(X, Y)$ is isomorphic to $H_{n-2}(S)$, where $n = \dim X$. Observing that $\dim V = n - 2$ and applying (c), we obtain $H_{n-2}(S) = 0$. Thus e^* is injective and (i) implies

$$(j) \quad w_2(\zeta) = v.$$

The vector bundle ζ , being algebraic, has a constant rank on each irreducible component of X . It follows that there exists an algebraic vector bundle ϵ on X such that the restriction of ϵ to each irreducible component of X is algebraically trivial and $\zeta \oplus \epsilon$ has constant rank, say, r on X . The line bundle $\lambda = \wedge^r(\zeta \oplus \epsilon)$ is algebraic [11, Proposition 12.1.8] and hence the vector bundle $\xi = \zeta \oplus \epsilon \oplus \lambda \oplus \lambda \oplus \lambda$ is also algebraic. Since $w_1(\lambda) = w_1(\zeta \oplus \epsilon)$ [21, p. 246], we have $w_1(\xi) = 0$ and, in view of (j), $w_2(\xi) = v$. Thus the proof is complete. \square