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classes in codimension 2 by Zariski closed subsets. More precisely, we have the following result.

THEOREM 1.6. *Let M be a compact orientable smooth manifold of dimension at least 5 and let G be a subgroup of $H^2(M;\mathbf{Z}/2)$. Then the following conditions are equivalent:*

(a) There exist a nonsingular real algebraic variety X and a diffeomorphism $\varphi\colon X \to M$ such that $\varphi^*(G) = H^2_{\mathrm{alg}}(X;\mathbf{Z}/2)$.

(b) $w_2(M) \in G \subseteq W^2(M)$, where $w_2(M)$ is the second Stiefel-Whitney class of M.

Proof. See [13].

Another application concerns the problem of approximation of smooth curves (that is, one-dimensional smooth submanifolds) by algebraic curves. First recall that a compact smooth submanifold N of a nonsingular real algebraic variety X is said to *admit an algebraic approximation* in X if for each neighborhood $\mathcal{U}$ of the inclusion map $N \hookrightarrow X$ (in the $\mathcal{C}^\infty$ topology on the set $\mathcal{C}^\infty(N,X)$ of smooth maps from N into X), there exists a smooth embedding $e\colon N \to X$ such that e is in $\mathcal{U}$ and $e(N)$ is a nonsingular Zariski closed subset of X.

THEOREM 1.7. *Let X be a compact nonsingular real algebraic variety of dimension 3 and let C be a compact smooth curve in X. Then C admits an algebraic approximation in X if and only if the $\mathbf{Z}/2$-homology class represented by C is in $H_1^{\mathrm{alg}}(X;\mathbf{Z}/2)$.*

The proof of Theorem 1.7 will be given elsewhere. Under the extra assumption that C is connected and homologous to the union of finitely many nonsingular real algebraic curves in X the theorem is proved in [4].

2. PROOF OF THE GROTHENDIECK FORMULA

We shall use homology and cohomology groups with coefficients exclusively in $\mathbf{Z}/2$ and therefore we shall suppress the coefficient group in our notation.

For any continuous map $f\colon (X,A) \to (Y,B)$ between pairs of topological spaces, we let

$$f_*\colon H_k(X,A) \to H_k(Y,B)\,, \quad f^*\colon H^k(Y,B) \to H^k(X,A)$$

denote the induced homomorphisms.

For the convenience of the reader we shall now review some facts from topology. Let B be a paracompact topological space and let $\xi = (E, \pi, B)$ be a real vector bundle of rank k on B. Let $s_0\colon B \to E$ be the zero section of ξ, that is, $s_0(x) = 0_x$ for all x in B, where 0_x is the zero vector in the fiber $E_x = \pi^{-1}(x)$. We set $0_E = s_0(B)$. Recall that the Thom class τ_ξ of ξ is a unique element of $H^k(E, E \setminus 0_E)$ such that for every point x in B, the homomorphism

$$H^k(E, E\setminus 0_E) \to H^k(E_x, E_x \setminus \{0_x\}) \cong \mathbf{Z}/2\,,$$

induced by the inclusion map $(E_x, E_x\setminus\{0_x\}) \hookrightarrow (E, E\setminus 0_E)$, sends τ_ξ to the generator of $\mathbf{Z}/2$ [24, Theorem 8.1] (the name "Thom class" is not used in [24]). For every nonnegative integer q, we have the Thom isomorphism

$$\varphi_q\colon H^q(B) \to H^{k+q}(E, E\setminus 0_E)$$
$$\varphi_q(v) = \pi^*(v) \cup \tau_\xi \quad \text{for all } v \text{ in } H^q(B)$$

[24, Definition 8.2].

If $s\colon B \to E$ is any continuous section of ξ and $\bar{s}\colon (B, B\setminus s^{-1}(0_E)) \to (E, E\setminus 0_E)$ is the map defined by s, then

(2.1) $$w_k(\xi) = i^*(\bar{s}^*(\tau_\xi))\,,$$

where $i\colon B = (B, \varnothing) \hookrightarrow (B, B\setminus s^{-1}(0_E))$ is the inclusion map. Indeed, let $j\colon E \hookrightarrow (E, E\setminus 0_E)$ be the inclusion map. Note that $H\colon E\times[0,1] \to (E, E\setminus 0_E)$, defined by $H(e,t) = (1-t)j(e) + t(\bar{s}\circ i\circ \pi)(e)$ for all (e,t) in $E\times[0,1]$, is a homotopy between j and $\bar{s}\circ i\circ\pi$. In particular, $j^* = (\bar{s}\circ i\circ\pi)^* = \pi^*\circ i^*\circ \bar{s}^*$, and hence

$$\pi^*(i^*(\bar{s}^*(\tau_\xi))) \cup \tau_\xi = j^*(\tau_\xi)\cup\tau_\xi = \tau_\xi \cup \tau_\xi\,,$$

where the last equality is the standard property of the cup product [26, p. 251, property 8]. Thus $\varphi_k(i^*(\bar{s}^*(\tau_\xi))) = \tau_\xi\cup\tau_\xi$. Now, (2.1) follows since $w_k(\xi) = \varphi_k^{-1}(\tau_\xi\cup\tau_\xi)$ [24, p. 91].

Let M be a smooth m-dimensional manifold and let N be a smooth n-dimensional submanifold of M. Assume that N is a closed subset of M. A tubular neighborhood of N in M is a smooth real vector bundle $\xi = (E, \pi, N)$ on N such that E is an open neighborhood of N in M and $0_E = N$ [20]. By the excision property, the inclusion map $e\colon (E, E\setminus N) \hookrightarrow (M, M\setminus N)$ induces an isomorphism

$$e^*: H^k(M, M\setminus N) \to H^k(E, E\setminus N),$$

where $k = m - n$. The Thom class τ_N^M of N in M is a unique element of $H^k(M, M\setminus N)$ such that $e^*(\tau_N^M) = \tau_\xi$. The Thom isomorphism yields

$$H^k(M, M\setminus N) \cong H^0(N).$$

Hence

(2.2) $$\tau_N^M \text{ generates } H^k(M, M\setminus N) \cong \mathbf{Z}/2,$$

provided N is connected. Assuming that N has exactly r connected components $N_1, \dots, N_r$, the inclusion maps $e_i\colon (M, M\setminus N) \hookrightarrow (M, M\setminus N_i)$ give rise to an isomorphism

$$t\colon \bigoplus_{i=1}^{r} H^k(M, M\setminus N_i) \to H^k(M, M\setminus N)$$

$$t(u_1, \dots, u_r) = e_1^*(u_1) + \cdots + e_r^*(u_r)$$

satisfying

(2.3) $$t(\tau_{N_1}^M, \dots, \tau_{N_r}^M) = \tau_N^M.$$

If $f\colon M \to P$ is a smooth map between smooth manifolds, transverse to a smooth submanifold Q of P (Q a closed subset of P) and with $N = f^{-1}(Q)$, then

(2.4) $$\bar{f}^*(\tau_Q^P) = \tau_N^M,$$

where $\bar{f}\colon (M, M\setminus N) \to (P, P\setminus Q)$ is the map defined by f. Indeed, after a homotopy, f looks like a vector bundle map between tubular neighborhoods of N and Q [20, p. 117, Theorem 6.7], and hence (2.4) follows from the definition of the Thom class.

Let Δ be the diagonal of $M \times M$,

$$\Delta = \{(x, y) \in M \times M \mid x = y\},$$

and let τ in $H^m(M \times M, (M \times M)\setminus\Delta)$ be the Thom class of Δ in $M \times M$. For every point x in M, the image of τ under the homomorphism

$$H^m(M \times M, (M \times M)\setminus\Delta) \to H^m(M, M\setminus\{x\}) \cong \mathbf{Z}/2$$

induced by the map $(M, M\setminus\{x\}) \to (M \times M, (M \times M)\setminus\Delta)$, $y \to (x, y)$, generates $\mathbf{Z}/2$ [24, Lemma 11.7]. Thus τ is the orientation class of M over $\mathbf{Z}/2$ in the terminology used in [26, p. 294]. For any pair (A, B) of subsets of M, $B \subseteq A$, and any integer q satisfying $0 \leq q \leq m$, let

$$\gamma_{A,B}\colon H_q(A, B) \to H^{m-q}(M \setminus B,\ M \setminus A)$$

be the homomorphism defined by

$$\gamma_{A,B}(a) = a \diagdown j^*_{A,B}(\tau)\,,$$

where $\diagdown$ is the slant product and

$$j_{A,B}\colon (A \times (M \setminus B), (A \times (M \setminus A)) \cup (B \times (M \setminus B))) \hookrightarrow (M \times M, (M \times M) \setminus \Delta)$$

is the inclusion map, cf. [26, p. 351]. If B is empty, we shall write γ_A instead of $\gamma_{A,\varnothing}$. The following naturality property is satisfied: if (A', B') is another pair of subsets of M, $B' \subseteq A'$, and $A \subseteq A'$, $B \subseteq B'$, then the diagram

(2.5)
$$\begin{array}{ccc} H_q(A, B) & \xrightarrow{\gamma_{A,B}} & H^{m-q}(M \setminus B, M \setminus A) \\ \downarrow & & \downarrow \\ H_q(A', B') & \xrightarrow{\gamma_{A',B'}} & H^{m-q}(M \setminus B', M \setminus A')\,, \end{array}$$

where the vertical homomorphisms are induced by the appropriate inclusion maps, is commutative [26, pp. 287, 289, 351]. Furthermore, if M is compact, then

(2.6)
$$\gamma_M = D_M^{-1}\,,$$

that is,

$$\gamma_M\colon H_q(M) \to H^{m-q}(M)$$

is the inverse of the Poincaré duality isomorphism

$$D_M\colon H^{m-q}(M) \to H_q(M)\,, \quad D_M(u) = u \cap [M]\,.$$

This follows from [26, p. 305, Theorem 12] and the fact that, in the notation of [26, p. 353, Lemma 15], θ is the identity map, provided $X = Y$, $G = \mathbf{Z}/2$.

We shall also make use of the following result.

PROPOSITION 2.7. *If M is compact and (A, B) is a compact polyhedral pair in M, then*

$$\gamma_{A,B}\colon H_q(A, B) \to H^{m-q}(M \setminus B,\ M \setminus A)$$

is an isomorphism.

Proof. We have the following diagram:

$$\begin{array}{ccc}
H_q(B) & \xrightarrow{\gamma_B} & H^{m-q}(M, M\setminus B) \\
\downarrow & & \downarrow \\
H_q(A) & \xrightarrow{\gamma_A} & H^{m-q}(M, M\setminus A) \\
\downarrow & & \downarrow \\
H_q(A,B) & \xrightarrow{\gamma_{A,B}} & H^{m-q}(M\setminus B, M\setminus A) \\
\downarrow & & \downarrow \\
H_{q-1}(B) & \xrightarrow{\gamma_B} & H^{m-q+1}(M, M\setminus B) \\
\downarrow & & \downarrow \\
H_{q-1}(A) & \xrightarrow{\gamma_A} & H^{m-q+1}(M, M\setminus A),
\end{array}$$

where the columns are parts of the long exact sequences for the pair (A, B) and the triple $(M, M\setminus B, M\setminus A)$. By (2.5) and [26, p. 287, property 3, and p. 351], the diagram is commutative. It is proved in [26, p. 351, Lemma 14] that γ_A and γ_B are isomorphisms for q and $q-1$. In view of the five lemma, $\gamma_{A,B}$ is also an isomorphism. □

After this preparation, we are ready to prove an auxiliary result relating homology and cohomology of real algebraic varieties. Let X be a compact n-dimensional nonsingular real algebraic variety and let V be a d-dimensional Zariski closed subset of X. By Theorem 1.1, V is a compact polyhedron and hence

$$\gamma_V: H_d(V) \to H^c(X, X\setminus V),$$

where $c = n - d$, is an isomorphism in view of Proposition 2.7. For our purposes it is important to give a characterization of $\gamma_V([V])$. Set $S = \mathrm{Sing}(V)$ and let

$$i: (X\setminus S, (X\setminus S)\setminus(V\setminus S)) \hookrightarrow (X, X\setminus V),\ j: X \hookrightarrow (X, X\setminus V)$$

be the inclusion maps (of course, $X\setminus V = (X\setminus S)\setminus(V\setminus S)$). Since $V\setminus S$ is a d-dimensional nonsingular Zariski closed subset of $X\setminus S$, the Thom class $\tau_{V\setminus S}^{X\setminus S}$ in $H^c(X\setminus S, (X\setminus S)\setminus(V\setminus S))$ is defined.

PROPOSITION 2.8. *There exists a unique element* τ_V^X *in* $H^c(X, X\setminus V)$ *such that*

$$i^*(\tau_V^X) = \tau_{V\setminus S}^{X\setminus S}.$$

Furthermore,

$$\tau_V^X = \gamma_V([V]) \quad \textit{and} \quad D_X(j^*(\tau_V^X)) = [V]_X.$$

Proof. We shall first prove $i^*(\gamma_V([V])) = \tau_{V\setminus S}^{X\setminus S}$. The smooth manifold $V \setminus S$ is a semialgebraic set and therefore has finitely many connected components, say $N_1, \ldots, N_r$ [11, p. 35]. If V_i is the closure of N_i in V and $S_i = V_i \cap S$, then $N_i = V_i \setminus S_i$. Note that V_i and S_i are compact semialgebraic subsets of V [8, p. 61 or 11, p. 27]. By (2.5), we have the following commutative diagram:

$$\begin{array}{ccccc}
H_d(V) & \xrightarrow{\varphi} & H_d(V,S) & \xleftarrow{\alpha} & \bigoplus_{i=1}^r H_d(V_i,S_i) \\
\gamma_V \downarrow & & \gamma_{V,S} \downarrow & & \bigoplus_{i=1}^r \gamma_{V_i,S_i} \downarrow \\
H^c(X,X\setminus V) & \xrightarrow{i^*} & H^c(X\setminus S,(X\setminus S)\setminus(V\setminus S)) & \xleftarrow{\beta} & \bigoplus_{i=1}^r H^c(X\setminus S_i,(X\setminus S_i)\setminus N_i),
\end{array}$$

where φ is induced by the appropriate inclusion map, whereas

$$\alpha(a_1, \ldots, a_r) = \alpha_1(a_1) + \cdots + \alpha_r(a_r),$$
$$\beta(u_1, \ldots, u_r) = \beta_1(u_1) + \cdots + \beta_r(u_r),$$

with

$$\alpha_i \colon H_d(V_i, S_i) \to H_d(V, S)$$
$$\beta_i \colon H^c(X \setminus S_i, (X \setminus S_i) \setminus N_i) \to H^c(X \setminus S, (X \setminus S) \setminus (V \setminus S))$$

induced by the inclusion maps.

Since $N_1, \ldots, N_r$ are the connected components of the smooth manifold $V \setminus S$, we have another commutative diagram:

$$\begin{array}{ccc}
H^c(X\setminus S,(X\setminus S)\setminus(V\setminus S)) & \xleftarrow{\beta} & \bigoplus_{i=1}^r H^c(X\setminus S_i,(X\setminus S_i)\setminus N_i) \\
t \uparrow & & \bigoplus_{i=1}^r \psi_i \downarrow \\
\bigoplus_{i=1}^r H^c(X\setminus S,(X\setminus S)\setminus N_i) & \xleftarrow{id} & \bigoplus_{i=1}^r H^c(X\setminus S,(X\setminus S)\setminus N_i),
\end{array}$$

where

$$\psi_i \colon H^c(X \setminus S_i, (X \setminus S_i) \setminus N_i) \to H^c(X \setminus S, (X \setminus S) \setminus N_i)$$

is the homomorphism induced by the appropriate inclusion map and t is the isomorphism of (2.3). It follows from the definiton of the Thom class that

$$(a) \qquad \psi_i(\tau_{N_i}^{X\setminus S_i}) = \tau_{N_i}^{X\setminus S}.$$

Hence, in view of (2.2), ψ_i is an isomorphism of cyclic groups isomorphic to $\mathbf{Z}/2$. Applying (2.3) and (a), we get

$$(b) \qquad \beta(\tau_{N_1}^{X\setminus S_1}, \dots, \tau_{N_r}^{X\setminus S_r}) = \tau_{V\setminus S}^{X\setminus S}.$$

Since, by Proposition 2.7, γ_{V_i,S_i} is an isomorphism, the group $H_d(V_i, S_i)$ is isomorphic to $\mathbf{Z}/2$; let a_i be its unique generator. Now, (a) and (b) imply

$$\gamma_{V,S}(\alpha(a_1, \dots, a_r)) = \tau_{V\setminus S}^{X\setminus S}.$$

Thus in order to verify $i^*(\gamma_V([V])) = \tau_{V\setminus S}^{X\setminus S}$ it suffices to prove

$$(c) \qquad \alpha(a_1, \dots, a_r) = \varphi([V]),$$

which can be done as follows.

Let $\Phi\colon |K| \to V$ be a semialgebraic triangulation of V compatible with $\{V_1, \dots, V_r, S_1, \dots, S_r\}$ (Theorem 1.1). Denote by c_i the chain which is the sum of all d-simplices of K whose images under Φ are contained in V_i. Since $N_i = V_i \setminus S_i$ is a smooth d-dimensional manifold, it follows that every open $(d-1)$-simplex σ of K with $\Phi(\sigma)$ contained in N_i is a face of exactly two d-simplices of K. Thus c_i represents a nonzero homology class in $H_d(V_i, S_i) \cong \mathbf{Z}/2$; in other words, c_i represents a_i. On the other hand, $c_1 + \cdots + c_r$ is the sum of all d-simplices of K and therefore it is a cycle representing the fundamental class $[V]$ in $H_d(V)$. Hence (c) follows and $i^*(\gamma_V([V])) = \tau_{V\setminus S}^{X\setminus S}$ is proved.

Let us observe that i^* is injective. Indeed, there is an exact sequence

$$\cdots \to H^c(X, X\setminus S) \to H^c(X, X\setminus V) \to H^c(X\setminus S, X\setminus V) \to \cdots$$

corresponding to the triple $(X, X\setminus S, X\setminus V)$. By Proposition 2.7, $\gamma_S\colon H_d(S) \to H^c(X, X\setminus S)$ is an isomorphism. Since $\dim S < d$, we obtain $H_d(S) = 0$, which implies $H^c(X, X\setminus S) = 0$. Hence i^* is injective as asserted.

Thus $\tau_V^X = \gamma_V([V])$ is a unique element of $H^c(X, X\setminus V)$ satisfying $i^*(\tau_V^X) = \tau_{V\setminus S}^{X\setminus S}$.

It remains to prove $D_X(j^*(\tau_V^X)) = [V]_X$. By (2.5), we have the following commutative diagram:

$$\begin{array}{ccc} H_d(V) & \xrightarrow{e_*} & H_d(X) \\ \gamma_V \downarrow & & \gamma_X \downarrow \\ H^c(X, X\setminus V) & \xrightarrow{j^*} & H^c(X), \end{array}$$

where $e\colon V \hookrightarrow X$ is the inclusion map. In view of (2.6), γ_X is the inverse of D_X and we obtain $D_X(j^*(\tau_V^X)) = e_*([V]) = [V]_X$. Thus the proof is complete. □

We shall now recall a purely algebraic result. Definitions of algebraic terms not explained here can all be found in [23]. Given a ring R (commutative with identity), we let $K_0(R)$ denote the Grothendieck group of finitely generated projective R-modules. If S is a multiplicatively closed subset of R and $S^{-1}R$ denotes the ring of fractions of R with denominators in S, then the canonical ring homomorphism $j_S\colon R \to S^{-1}R$, $j_S(r) = r/1$, induces a group homomorphism $K_0(R) \to K_0(S^{-1}R)$. Assuming that R is a regular ring of finite Krull dimension, every finitely generated R-module has a finite projective resolution [23, p. 208]. The last fact allows one to apply [6, p. 453, Proposition 2.1, p. 492, Proposition 6.1], which yields the result we require: the homomorphism $K_0(R) \to K_0(S^{-1}R)$ is surjective, provided that R is a regular ring of finite Krull dimension (this also easily follows from [23, p. 210, Exercise 4]).

To make use of this result we need some algebraic properties of the ring $\mathcal{R}(X)$ of regular functions on a real algebraic variety X. Suppose that X is a Zariski locally closed subset of $\mathbf{R}^n$ and let $\mathcal{P}(X)$ be the ring of polynomial functions from X into $\mathbf{R}$ ($f\colon X \to \mathbf{R}$ is a polynomial function if for some polynomial P in $\mathbf{R}[T_1, \dots, T_n]$, one has $f(x) = P(x)$ for all x in X). Clearly, $\mathcal{P}(X)$ is a finitely generated $\mathbf{R}$-algebra and thus a Noetherian ring [23, p. 11]. Furthermore, the Krull dimension of $\mathcal{P}(X)$ is equal to $\dim X$ [11, p. 50]. Recall that $\mathcal{R}(X)$ consists of all functions of the form f/g, where f, g are in $\mathcal{P}(X)$ and $g^{-1}(0) = \varnothing$. In other words, $\mathcal{R}(X)$ is the ring of fractions of $\mathcal{P}(X)$ with denominators in the set $\{g \in \mathcal{P}(X) \mid g^{-1}(0) = \varnothing\}$. It follows that $\mathcal{R}(X)$ is a Noetherian ring of Krull dimension $\dim X$ [23, p. 81]. Obviously, for every point x in X,

$$m_x = \{f \in \mathcal{R}(X) \mid f(x) = 0\}$$

is a maximal ideal of $\mathcal{R}(X)$ and each maximal ideal of $\mathcal{R}(X)$ is equal to m_x for some x. The localization $\mathcal{R}(X)_x$ of $\mathcal{R}(X)$ with respect to m_x is a Noetherian local ring of Krull dimension not exceeding $\dim X$ [23, p. 81]. A point x in X is nonsingular if and only if the local ring $\mathcal{R}(X)_x$ is regular of Krull dimension $\dim X$ [11, p. 67]. In particular, the ring $\mathcal{R}(X)$ is regular of finite Krull dimension, provided X is nonsingular. Given a Zariski open subset U of X, the subset

$$S(U) = \{g \in \mathcal{R}(X) \mid g^{-1}(0) \subseteq X \setminus U\}$$

of $\mathcal{R}(X)$ is multiplicatively closed. Since $\mathcal{R}(U) = S(U)^{-1}\mathcal{R}(X)$, it follows from the facts reviewed above that the group homomorphism

$$K_0(\mathcal{R}(X)) \to K_0(\mathcal{R}(U)), \tag{2.9}$$

induced by the restriction ring homomorphism $\mathcal{R}(X) \to \mathcal{R}(U)$, $f \to f|_U$, is surjective, assuming X is nonsingular.

PROPOSITION 2.10. *Let X be a nonsingular real algebraic variety and let U be a Zariski open subset of X. For any algebraic vector bundle η on U, there exists an algebraic vector bundle ξ on X such that $\xi|_U$ and η are algebraically stably equivalent (that is, one can find algebraically trivial vector bundles ϵ_1 and ϵ_2 on U with the property that the bundles $(\xi|_U) \oplus \epsilon_1$ and $\eta \oplus \epsilon_2$ on U are algebraically isomorphic).*

Proof. Let Y be a real algebraic variety. For any algebraic vector bundle ζ on Y, let $\Gamma(\zeta)$ denote the $\mathcal{R}(Y)$-module of algebraic global sections of ζ. One readily proves that the correspondence $\zeta \to \Gamma(\zeta)$ establishes an equivalence of the category of algebraic vector bundles on Y with the category of finitely generated projective $\mathcal{R}(Y)$-modules [11, Proposition 12.1.12]. The proposition follows since (2.9) is surjective. □

Let Y be a real algebraic variety and let W be a Zariski closed subset of Y. Denote by $I_Y(W)$ the ideal of $\mathcal{R}(Y)$ consisting of all regular functions vanishing on W,

$$I_Y(W) = \{f \in \mathcal{R}(Y) \mid f(y) = 0 \text{ for all } y \text{ in } W\}.$$

The restriction homomorphism $\mathcal{R}(Y) \to \mathcal{R}(W)$, $f \to f|_W$, gives rise, for each point y in W, to a ring epimorphism $\mathcal{R}(Y)_y \to \mathcal{R}(W)_y$, whose kernel is equal to the ideal $I_Y(W)\mathcal{R}(Y)_y$ of $\mathcal{R}(Y)_y$. In particular, the quotient ring $\mathcal{R}(Y)_y/I_Y(W)\mathcal{R}(Y)_y$ is isomorphic to $\mathcal{R}(W)_y$. Therefore if y in W is a nonsingular point of Y and $k = \dim Y - \dim W$, then given elements $f_1, \dots, f_k$ of $I_Y(W)$, the following conditions are equivalent:

(i) $I_Y(W)\mathcal{R}(Y)_y = (f_1, \dots, f_k)\mathcal{R}(Y)_y$ and y is a nonsingular point of W,

(ii) $I_Y(W)\mathcal{R}(Y)_y = (f_1, \dots, f_k)\mathcal{R}(Y)_y$ and there exist elements $f_{k+1}, \dots, f_{k+d}$ of $\mathcal{R}(Y)$, $d = \dim W$, such that $f_1, \dots, f_{k+d}$ generate the unique maximal ideal of the local ring $\mathcal{R}(Y)_y$,

(iii) the map $(f_1, \dots, f_k)\colon Y \setminus \operatorname{Sing}(Y) \to \mathbf{R}^k$ is transverse to 0 at y and $W \cap H = f_1^{-1}(0) \cap \dots \cap f_k^{-1}(0) \cap H$, where H is a Zariski open neighborhood of y in $Y \setminus \operatorname{Sing}(Y)$.

Indeed, the equivalence of (i) and (ii) is a consequence of [23, p. 169, Proposition 1.10]. Furthermore, $f_1, \dots, f_{k+d}$ generate the maximal ideal of $\mathcal{R}(Y)_y$ if and only if there exists a neighborhood N of y in $Y \setminus \operatorname{Sing}(Y)$ such that the restriction of $(f_1, \dots, f_{k+d})$ to N is a local coordinate system for the smooth manifold $Y \setminus \mathrm{Sing}(Y)$ [11, pp. 66, 67]. Hence the equivalence of (ii) and (iii) easily follows.

It also follows from [23, p. 169, Proposition 1.10] that $I_Y(W)\mathcal{R}(Y)_y$ is generated by k elements, provided y in W is a nonsingular point of Y and of W.

We shall freely use the facts just reviewed.

Proof of Theorem 1.5. By assumption, $D_X(v) = [V]_X$, where V is a Zariski closed subset of X with $\dim X - \dim V = 2$. If $V_1, \dots, V_p$ are the irreducible components of V of dimension $\dim V$, then $[V]_X = [V_1]_X + \cdots + [V_p]_X$, and hence it suffices to prove the theorem assuming that V is irreducible.

Let x_0 be a nonsingular point of V. Then the ideal $I_X(V)\mathcal{R}(X)_{x_0}$ of the ring $\mathcal{R}(X)_{x_0}$ can be generated by two elements; we choose generators a_1, a_2 that belong to $I_X(V)$. Hence there exists a Zariski open neighborhood U of x_0 in X such that the ideal $I_X(V)\mathcal{R}(U)$ of the ring $\mathcal{R}(U)$ is generated by a_1 and a_2. This implies

(*a*) $$I_X(V)\mathcal{R}(U)_x = (a_1, a_2)\mathcal{R}(U)_x \text{ for all } x \text{ in } U.$$

Since $\mathrm{Sing}(V)$ is Zariski closed in V, shrinking U if necessary, we may assume that $U \cap \mathrm{Sing}(V) = \varnothing$. Hence from (*a*), we obtain

(*b*) the map $(a_1, a_2)\colon U \to \mathbf{R}^2$ is transverse to 0 in $\mathbf{R}^2$ at each point x in $U \cap V$.

Setting $S = V \setminus (U \cap V)$, we have $\mathrm{Sing}(V) \subseteq S$ and, by virtue of irreducibility of V,

(*c*) $$\dim S < \dim V.$$

Let $Y = X \setminus S$ and $W = V \setminus S$. Then Y is a Zariski open subset of X and W is a Zariski closed subset of Y, with $\dim Y - \dim W = 2$.

CLAIM. *There exist an algebraic vector bundle $\eta = (E, \pi, Y)$ on Y and an algebraic section $s\colon Y \to E$ of η such that η is of rank 2, $W = s^{-1}(0_E)$, and s is transverse to 0_E.*

We prove the claim as follows. Choose a regular function b in $\mathcal{R}(Y)$ with $b^{-1}(0) = W$. Set $b_k = a_k|_Y$ for $k = 1, 2$, and define a map $F\colon Y \times \mathbf{R}^2 \to \mathbf{R}^2$

by

$$F(y,t) = F_t(y) = (b_1(y) + t_1 b(y)^2,\ b_2(y) + t_2 b(y)^2)$$

for all y in Y and $t = (t_1, t_2)$ in $\mathbf{R}^2$.

We assert that F is transverse to 0 in $\mathbf{R}^2$. Indeed, suppose $F(y,t) = 0$ for some (y,t) in $Y \times \mathbf{R}^2$. If y is not in W, then the assertion holds since it suffices to consider the partial derivatives with respect to t_1 and t_2. If y is in W, then (b) implies that $F_t \colon Y \to \mathbf{R}^2$ is transverse to 0 in $\mathbf{R}^2$ at y, which means that the assertion also holds in this case. Hence the assertion is proved.

It follows from the assertion and a standard transversality theorem [20, p. 79, Theorem 2.7] that there exists a point t in $\mathbf{R}^2$ for which the map

$$F_t = (f_1, f_2) \colon Y \to \mathbf{R}^2$$

is transverse to 0 in $\mathbf{R}^2$. Since f_1 and f_2 are in $I_Y(W)$ and W is nonsingular, we get

$$I_Y(W)\mathcal{R}(Y)_y = (f_1, f_2)\mathcal{R}(Y)_y$$

for all y in W. Hence for each point y in W, one can find a Zariski open neighborhood G_y of y in Y with

$$I_Y(W)\mathcal{R}(G_y) = (f_1, f_2)\mathcal{R}(G_y).$$

In particular, $W \cap G_y = f_1^{-1}(0) \cap f_2^{-1}(0) \cap G_y$. Taking G to be the union of the G_y for y in W, we get $W = f_1^{-1}(0) \cap f_2^{-1}(0) \cap G$, which implies

$$(d) \qquad f_1^{-1}(0) \cap f_2^{-1}(0) = W \cup W',$$

where W' is a subset of Y disjoint from W. Clearly, W' is contained in $Y \setminus G$. Since $W \cup W'$ and $Y \setminus G$ are Zariski closed subsets of Y, and $W' = (W \cup W') \cap (Y \setminus G)$, it follows that W' is also Zariski closed in Y. The transversality of $(f_1, f_2) \colon Y \to \mathbf{R}^2$ to 0 in $\mathbf{R}^2$ together with (d) imply

$$(e) \qquad I_Y(W \cup W')\mathcal{R}(Y)_y = (f_1, f_2)\mathcal{R}(Y)_y \text{ for all } y \text{ in } Y.$$

Choosing regular functions ψ_1 and ψ_2 in $\mathcal{R}(Y)$ with $\psi_1^{-1}(0) = W$ and $\psi_2^{-1}(0) = W'$ (this is possible since W and W' are Zariski closed in Y), we see that $\psi_1\psi_2$ belongs to $I_Y(W \cup W')$ and hence

$$\psi_1\psi_2 = h_1 f_1 + h_2 f_2$$

for some regular functions h_1 and h_2 in $\mathcal{R}(Y)$ (the last assertion can easily be deduced directly from (e), but, anyhow, it is also a consequence of (e) and [23, p. 93, Rule 1.1]).

Let $\mathbf{M}_2(\mathbf{R})$ denote the set of all real 2×2 matrices (identified with $\mathbf{R}^4$ and regarded as a real algebraic variety). Consider regular maps $g_{21}\colon U_1 = Y \setminus W \to \mathbf{M}_2(\mathbf{R})$ and $g_{12}\colon U_2 = Y \setminus W' \to \mathbf{M}_2(\mathbf{R})$ defined by

$$g_{21} = \begin{bmatrix} f_1\psi_2/\psi_1 & -h_2/\psi_1^2 \\ f_2\psi_2/\psi_1 & h_1/\psi_1^2 \end{bmatrix}, \quad g_{12} = \begin{bmatrix} h_1/\psi_2^2 & h_2/\psi_2^2 \\ -f_2\psi_1/\psi_2 & f_1\psi_1/\psi_2 \end{bmatrix}.$$

For each point y in $U_1 \cap U_2$, the matrices $g_{12}(y)$ and $g_{21}(y)$ are invertible and $g_{12}(y)g_{21}(y)$ is the identity matrix. Define

$$E = \{(y, v_1, v_2) \in Y \times \mathbf{R}^2 \times \mathbf{R}^2 \mid v_1 = g_{12}(y)\, v_2 \text{ if } y \in U_2 \text{ and } v_2 = g_{21}(y)\, v_1 \text{ if } y \in U_1\}$$

and $\pi\colon E \to Y$, $\pi(y, v_1, v_2) = y$. Since $\{U_1, U_2\}$ is a Zariski open cover of Y, it follows that E is a Zariski closed subset of $Y \times \mathbf{R}^2 \times \mathbf{R}^2$. Clearly, π is a regular map and, for each point y in Y, the fiber $E_y = \pi^{-1}(y)$ is a vector subspace of $\{y\} \times \mathbf{R}^2 \times \mathbf{R}^2$. Furthermore, the map

$$U_k \times \mathbf{R}^2 \to \pi^{-1}(U_k),\ (y, v) \to (y, g_{1k}(y) \cdot v,\ g_{2k}(y) \cdot v)$$

is biregular for $k = 1, 2$, where $g_{kk}(y)$ is the identity matrix. Thus $\eta = (E, \pi, Y)$ is an algebraic vector bundle of rank 2 on Y. The map $s\colon Y \to E$

$$s(y) = (y, (\psi_1(y), 0),\ (f_1(y)\psi_2(y),\ f_2(y)\psi_2(y)))$$

is an algebraic section of η with $s^{-1}(0_E) = W$. On U_2 the section s is represented by $(f_1, f_2)\colon U_2 \to \mathbf{R}^2$, and therefore s is transverse to 0_E. Hence the claim is proved.

Let $\bar{s}\colon (Y, Y \setminus W) \to (E, E \setminus 0_E)$ be the map defined by s and let $\ell\colon Y \hookrightarrow (Y, Y \setminus W)$ be the inclusion map. In view of (2.1), we have $w_2(\eta) = \ell^*(\bar{s}^*(\tau_\eta))$, while (2.4) yields $\bar{s}^*(\tau_\eta) = \tau_W^Y$. It follows that

$$(f) \qquad w_2(\eta) = \ell^*(\tau_W^Y).$$

If $i\colon (Y, Y \setminus W) \hookrightarrow (X, X \setminus V)$, $j\colon X \hookrightarrow (X, X \setminus V)$, and $e\colon Y \hookrightarrow X$ are the inclusion maps, then the diagram

$$\begin{array}{ccc} H^2(X, X \setminus V) & \xrightarrow{\ i^*\ } & H^2(Y, Y \setminus W) \\ {\scriptstyle j^*}\downarrow & & \downarrow{\scriptstyle \ell^*} \\ H^2(X) & \xrightarrow{\ e^*\ } & H^2(Y) \end{array}$$

is commutative.

Since $W \subseteq V \setminus \mathrm{Sing}(V)$, Proposition 2.8 yields

$$(g) \qquad i^*(\tau_V^X) = \tau_W^Y, \qquad j^*(\tau_V^X) = v.$$

By combining (*d*) and (*e*), we get

$$(h) \qquad w_2(\eta) = \ell^*(i^*(\tau_V^X)) = e^*(j^*(\tau_V^X)) = e^*(v).$$

Proposition 2.10 implies that there exists an algebraic vector bundle ζ on X, whose restriction to Y is algebraically stably equivalent to η. In particular, $w_2(\eta) = w_2(\zeta \mid Y) = e^*(w_2(\zeta))$, and hence applying (*h*), we get

$$(i) \qquad e^*(v) = e^*(w_2(\zeta)).$$

Note that e^* is injective. Indeed, there is an exact sequence

$$H^2(X, Y) \longrightarrow H^2(X) \xrightarrow{e^*} H^2(Y).$$

Since $S = X \setminus Y$ is Zariski closed in X, by Theorem 1.1 and Proposition 2.7, $H^2(X, Y)$ is isomorphic to $H_{n-2}(S)$, where $n = \dim X$. Observing that $\dim V = n - 2$ and applying (*c*), we obtain $H_{n-2}(S) = 0$. Thus e^* is injective and (*i*) implies

$$(j) \qquad w_2(\zeta) = v.$$

The vector bundle ζ, being algebraic, has a constant rank on each irreducible component of X. It follows that there exists an algebraic vector bundle ϵ on X such that the restriction of ϵ to each irreducible component of X is algebraically trivial and $\zeta \oplus \epsilon$ has constant rank, say, r on X. The line bundle $\lambda = \Lambda^r(\zeta \oplus \epsilon)$ is algebraic [11, Proposition 12.1.8] and hence the vector bundle $\xi = \zeta \oplus \epsilon \oplus \lambda \oplus \lambda \oplus \lambda$ is also algebraic. Since $w_1(\lambda) = w_1(\zeta \oplus \epsilon)$ [21, p. 246], we have $w_1(\xi) = 0$ and, in view of (*j*), $w_2(\xi) = v$. Thus the proof is complete. □