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EXERCISE 11. Prove that the action of S_∞ on LO is continuous and minimal (that is, the orbit of each linear order is everywhere dense in LO).

Recall that a linear order $\prec$ is called *dense* if it has no gaps. A dense linear order without least and greatest elements is said to be of type η. The collection LO_η of all linear orders of type η on $\mathbf{Z}$ can be identified with the factor space $S_\infty/\mathrm{Aut}(\prec)$ through the correspondence $\sigma \mapsto {}^\sigma\!\prec$. Here $\prec$ is some chosen linear order of type η on $\mathbf{Z}$ and $\mathrm{Aut}(\prec)$ stands for the group of order-preserving self-bijections of $(\mathbf{Z}, \prec)$, acting on the space of orders in a natural way: $(x \,{}^\sigma\!\prec y) \Leftrightarrow \sigma^{-1}x \prec \sigma^{-1}y$.

EXERCISE 12. Show that under the above identification the uniform structure on LO_η, induced from the compact space LO, is the finest uniform structure making the quotient map $S_\infty \to S_\infty/\mathrm{Aut}(\prec) \cong \mathrm{LO}_\eta$ right uniformly continuous.

Let now X be a compact S_∞-space. The topological subgroup $\mathrm{Aut}(\prec)$ of S_∞ has a fixed point in X, say x' (Exercise 10). The mapping $S_\infty \ni \sigma \mapsto \sigma(x') \in X$ is constant on the left $\mathrm{Aut}(\prec)$-cosets and thus gives rise to a mapping $\varphi\colon \mathrm{LO}_\eta \to X$. Using Exercise 12, it is easy to see that φ is right uniformly continuous and thus extends to a morphism of S_∞-spaces $\mathrm{LO} \to X$. We have established the following result.

THEOREM 6 (Glasner and Weiss [Gl-W]). *The compact space* LO *forms the universal minimal* S_∞*-space.*

6.5 THE URYSOHN METRIC SPACE

The *universal Urysohn metric space* $\mathbf{U}$ [Ur] is determined uniquely (up to an isometry) by the following conditions:

(i) $\mathbf{U}$ is a complete separable metric space;

(ii) $\mathbf{U}$ is ω-homogeneous, that is, every isometry between two finite subspaces of $\mathbf{U}$ extends to an isometry of $\mathbf{U}$;

(iii) $\mathbf{U}$ contains an isometric copy of every separable metric space.

A probabilistic description of this space was given by Vershik [Ver]: the completion of the space of integers equipped with a 'sufficiently random' metric is almost surely isometric to $\mathbf{U}$.

The group of isometries $\mathrm{Iso}(\mathbf{U})$ with the compact-open topology is a Polish (complete metric separable) topological group, which also possesses a universality property: it contains an isomorphic copy of every separable metric group [Usp]. See also [Gr3].

Using concentration of measure, one can prove that the group Iso$(\mathbf{U})$ is extremely amenable. The Ramsey–Dvoretzky–Milman property leads to the following Ramsey-type result:

Let F be a finite metric space, and let all isometric embeddings of F into $\mathbf{U}$ be coloured using finitely many colours. Then for every finite metric space G and every $\varepsilon > 0$ there is an isometric copy $G' \subset \mathbf{U}$ of G such that all isometric embeddings of F into $\mathbf{U}$ that factor through G are monochromatic to within ε.

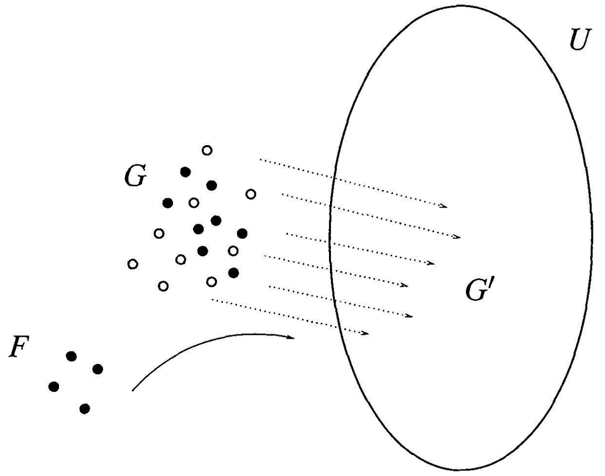

FIGURE 5
A Ramsey-type result for metric spaces

Here we say that a set A is *monochromatic to within* ε if there is a monochromatic set A' at a Hausdorff distance $< \varepsilon$ from A. In our case, the Hausdorff distance is formed with regard to the uniform metric on $\mathbf{U}^F$.

One can also obtain similar results, for example, for the separable Hilbert space ℓ_2 and for the unit sphere $\mathbf{S}^\infty$ in ℓ_2 [P3].

7. CONCENTRATION TO A NON-TRIVIAL SPACE

Let f be a Borel measurable real-valued function on an *mm*-space $X = (X, d, \mu)$. A number $M = M_f$ is called a *median* (or *Lévy mean*) of f if both $f^{-1}[M, +\infty)$ and $f^{-1}(-\infty, M]$ have measure $\geq \frac{1}{2}$.

EXERCISE 13. Show that the median M_f always exists, though it need not be unique.