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EXERCISE 11. Prove that the action of S_∞ on LO is continuous and minimal (that is, the orbit of each linear order is everywhere dense in LO).

Recall that a linear order \prec is called *dense* if it has no gaps. A dense linear order without least and greatest elements is said to be of type η . The collection LO_η of all linear orders of type η on \mathbf{Z} can be identified with the factor space $S_\infty/\text{Aut}(\prec)$ through the correspondence $\sigma \mapsto \sigma \prec$. Here \prec is some chosen linear order of type η on \mathbf{Z} and $\text{Aut}(\prec)$ stands for the group of order-preserving self-bijections of (\mathbf{Z}, \prec) , acting on the space of orders in a natural way: $(x \sigma \prec y) \Leftrightarrow \sigma^{-1}x \prec \sigma^{-1}y$.

EXERCISE 12. Show that under the above identification the uniform structure on LO_η , induced from the compact space LO, is the finest uniform structure making the quotient map $S_\infty \rightarrow S_\infty/\text{Aut}(\prec) \cong LO_\eta$ right uniformly continuous.

Let now X be a compact S_∞ -space. The topological subgroup $\text{Aut}(\prec)$ of S_∞ has a fixed point in X , say x' (Exercise 10). The mapping $S_\infty \ni \sigma \mapsto \sigma(x') \in X$ is constant on the left $\text{Aut}(\prec)$ -cosets and thus gives rise to a mapping $\varphi: LO_\eta \rightarrow X$. Using Exercise 12, it is easy to see that φ is right uniformly continuous and thus extends to a morphism of S_∞ -spaces $LO \rightarrow X$. We have established the following result.

THEOREM 6 (Glasner and Weiss [Gl-W]). *The compact space LO forms the universal minimal S_∞ -space.*

6.5 THE URYSOHN METRIC SPACE

The *universal Urysohn metric space* \mathbf{U} [Ur] is determined uniquely (up to an isometry) by the following conditions:

- (i) \mathbf{U} is a complete separable metric space;
- (ii) \mathbf{U} is ω -homogeneous, that is, every isometry between two finite subspaces of \mathbf{U} extends to an isometry of \mathbf{U} ;
- (iii) \mathbf{U} contains an isometric copy of every separable metric space.

A probabilistic description of this space was given by Vershik [Ver]: the completion of the space of integers equipped with a 'sufficiently random' metric is almost surely isometric to \mathbf{U} .

The group of isometries $\text{Iso}(\mathbf{U})$ with the compact-open topology is a Polish (complete metric separable) topological group, which also possesses

a universality property: it contains an isomorphic copy of every separable metric group [Usp]. See also [Gr3].

Using concentration of measure, one can prove that the group $\text{Iso}(\mathbf{U})$ is extremely amenable. The Ramsey–Dvoretzky–Milman property leads to the following Ramsey-type result:

Let F be a finite metric space, and let all isometric embeddings of F into \mathbf{U} be coloured using finitely many colours. Then for every finite metric space G and every $\varepsilon > 0$ there is an isometric copy $G' \subset \mathbf{U}$ of G such that all isometric embeddings of F into \mathbf{U} that factor through G are monochromatic to within ε .

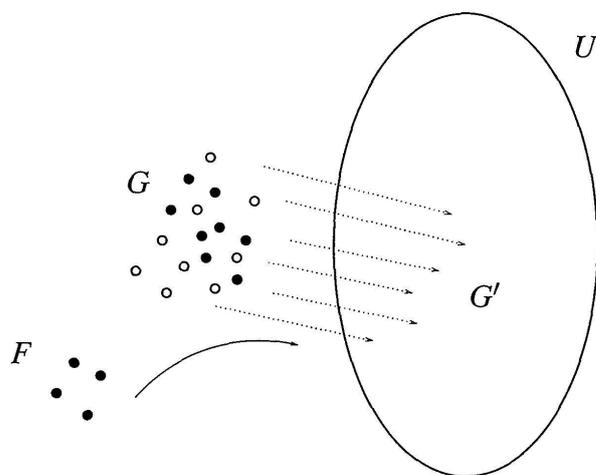


FIGURE 5

A Ramsey-type result for metric spaces

Here we say that a set A is *monochromatic to within ε* if there is a monochromatic set A' at a Hausdorff distance $< \varepsilon$ from A . In our case, the Hausdorff distance is formed with regard to the uniform metric on \mathbf{U}^F .

One can also obtain similar results, for example, for the separable Hilbert space ℓ_2 and for the unit sphere \mathbf{S}^∞ in ℓ_2 [P3].

7. CONCENTRATION TO A NON-TRIVIAL SPACE

Let f be a Borel measurable real-valued function on an *mm*-space $X = (X, d, \mu)$. A number $M = M_f$ is called a *median* (or *Lévy mean*) of f if both $f^{-1}[M, +\infty)$ and $f^{-1}(-\infty, M]$ have measure $\geq \frac{1}{2}$.

EXERCISE 13. Show that the median M_f always exists, though it need not be unique.