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with finitely many sets on each of which the given function f has oscillation $< \varepsilon$, and apply Ramsey's theorem. Use Remark 11.]

6.4 Extreme amenability and minimal flows

Corollary 8. *The group of orientation-preserving homeomorphisms of the closed unit interval,* $\mathrm{Homeo}_+(\mathbf{I})$*, equipped with the compact-open topology, is extremely amenable.*

Proof. Indeed, the extremely amenable group $\mathrm{Aut}(\mathbf{Q})$ admits a continuous monomorphism with a dense image into the group $\mathrm{Homeo}_+(\mathbf{I})$.

Remark 12. Thompson's group F consists of all piecewise-linear homeomorphisms of the interval whose points of non-smoothness are finitely many dyadic rational numbers, and the slopes of any linear part are powers of 2. (See [CFP].) It is a major open question in combinatorial group theory whether the Thompson group is amenable. Since F is everywhere dense in $\mathrm{Homeo}_+(\mathbf{I})$, our Corollary 8 does not contradict the possible amenability of F.

Using the extreme amenability of the topological groups $\mathrm{Aut}(\mathbf{Q})$ and $\mathrm{Homeo}_+(\mathbf{I})$, one is able to compute explicitly the universal minimal flows of some larger topological groups as follows.

Corollary 9. *The circle* $\mathbf{S}^1$ *forms the universal minimal* $\mathrm{Homeo}_+(\mathbf{S}^1)$*-space.*

Proof. Let $\theta \in \mathbf{S}^1$ be an arbitrary point. The isotropy subgroup St_θ of θ is isomorphic to $\mathrm{Homeo}_+(\mathbf{I})$. Because of that, whenever the topological group $\mathrm{Homeo}_+(\mathbf{S}^1)$ acts continuously on a compact space X, the subgroup St_θ has a fixed point, say $x' \in X$. The mapping $\mathrm{Homeo}_+(\mathbf{S}^1) \ni h \mapsto h(x') \in X$ is constant on the left St_θ-cosets and therefore gives rise to a continuous equivariant map $\mathrm{Homeo}_+(\mathbf{S}^1)/\mathrm{St}_\theta \cong \mathbf{S}^1 \to X$.

For the above results concerning groups $\mathrm{Aut}(\mathbf{Q})$, $\mathrm{Homeo}_+(\mathbf{I})$, and $\mathrm{Homeo}_+(\mathbf{S}^1)$, see [P1].

Now denote by LO the set of all linear orders on $\mathbf{Z}$, equipped with the (compact) topology induced from $\{0,1\}^{\mathbf{Z}\times\mathbf{Z}}$. The group S_∞ acts on LO by double permutations.

EXERCISE 11. Prove that the action of S_∞ on LO is continuous and minimal (that is, the orbit of each linear order is everywhere dense in LO).

Recall that a linear order $\prec$ is called *dense* if it has no gaps. A dense linear order without least and greatest elements is said to be of type η. The collection LO_η of all linear orders of type η on $\mathbf{Z}$ can be identified with the factor space $S_\infty/\mathrm{Aut}(\prec)$ through the correspondence $\sigma \mapsto {}^\sigma\!\prec$. Here $\prec$ is some chosen linear order of type η on $\mathbf{Z}$ and $\mathrm{Aut}(\prec)$ stands for the group of order-preserving self-bijections of $(\mathbf{Z}, \prec)$, acting on the space of orders in a natural way: $(x \,{}^\sigma\!\prec y) \Leftrightarrow \sigma^{-1}x \prec \sigma^{-1}y$.

EXERCISE 12. Show that under the above identification the uniform structure on LO_η, induced from the compact space LO, is the finest uniform structure making the quotient map $S_\infty \to S_\infty/\mathrm{Aut}(\prec) \cong \mathrm{LO}_\eta$ right uniformly continuous.

Let now X be a compact S_∞-space. The topological subgroup $\mathrm{Aut}(\prec)$ of S_∞ has a fixed point in X, say x' (Exercise 10). The mapping $S_\infty \ni \sigma \mapsto \sigma(x') \in X$ is constant on the left $\mathrm{Aut}(\prec)$-cosets and thus gives rise to a mapping $\varphi\colon \mathrm{LO}_\eta \to X$. Using Exercise 12, it is easy to see that φ is right uniformly continuous and thus extends to a morphism of S_∞-spaces $\mathrm{LO} \to X$. We have established the following result.

THEOREM 6 (Glasner and Weiss [Gl-W]). *The compact space* LO *forms the universal minimal S_∞-space.*

6.5 THE URYSOHN METRIC SPACE

The *universal Urysohn metric space* $\mathbf{U}$ [Ur] is determined uniquely (up to an isometry) by the following conditions:

(i) $\mathbf{U}$ is a complete separable metric space;

(ii) $\mathbf{U}$ is ω-homogeneous, that is, every isometry between two finite subspaces of $\mathbf{U}$ extends to an isometry of $\mathbf{U}$;

(iii) $\mathbf{U}$ contains an isometric copy of every separable metric space.

A probabilistic description of this space was given by Vershik [Ver]: the completion of the space of integers equipped with a 'sufficiently random' metric is almost surely isometric to $\mathbf{U}$.

The group of isometries $\mathrm{Iso}(\mathbf{U})$ with the compact-open topology is a Polish (complete metric separable) topological group, which also possesses