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**DEFINITION 11 [Gr1].** Say that a  $G$ -space  $X$  (in our agreed sense) has the *Ramsey–Dvoretzky–Milman property* if for every bounded uniformly continuous function  $f$  from  $X$  to a finite-dimensional Euclidean space, every  $\varepsilon > 0$ , and every finite  $F \subseteq X$ , there is a  $g \in G$  with the property

$$\text{Osc}(f|_{gF}) < \varepsilon.$$

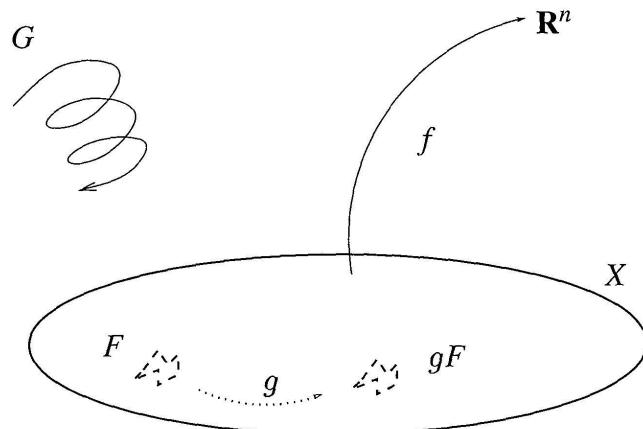


FIGURE 3  
The Ramsey–Dvoretzky–Milman property

**REMARK 9.** Equivalently,  $F$  can be assumed compact.

**COROLLARY 7.** For a topological group  $G$  the following are equivalent:

- (i)  $G$  is extremely amenable;
- (ii) every metric space  $X$  upon which  $G$  acts continuously and transitively by isometries has the R–D–M property;
- (iii) every homogeneous factor-space  $G/H$ , equipped with a left-invariant metric (or the left uniform structure), has the R–D–M property.

Next, we will discover two very important situations where the R–D–M property appears naturally.

## 6.2 DVORETZKY'S THEOREM

Here is the famous result.

**THEOREM** (Arieh Dvoretzky). For all  $\varepsilon > 0$  there is a constant  $c = c(\varepsilon) > 0$  such that for any  $n$ -dimensional normed space  $(X, \|\cdot\|_E)$  there is a subspace  $V$  of  $\dim V \geq c \log n$  and a Euclidean norm  $\|\cdot\|_2$  with  $\|x\|_2 \leq \|x\|_E \leq (1 + \varepsilon)\|x\|_2$  for all  $x \in V$ .

The studies of the phenomenon of concentration of measure were given a boost by Vitali Milman's new proof of the Dvoretzky theorem [M1], based on a suitable finite-dimensional approximation to the lemma which follows directly from results that we have previously stated:

**LEMMA (Milman).** *The pair  $(U(\mathcal{H}), S^\infty)$  has the R–D–M property, where  $S^\infty$  is the unit sphere of an infinite-dimensional Hilbert space  $\mathcal{H}$ .*

### 6.3 RAMSEY'S THEOREM

Let  $r$  be a positive natural number. By  $[r]$  one denotes the set  $\{1, 2, \dots, r\}$ . A *colouring* of a set  $X$  with  $r$  colours, or simply *r-colouring*, is any map  $\chi: X \rightarrow [r]$ . A subset  $A \subseteq X$  is *monochromatic* if for every  $a, b \in A$  one has  $\chi(a) = \chi(b)$ .

Put otherwise, a finite colouring of a set  $X$  is nothing but a partition of  $X$  into finitely many (disjoint) subsets.

Let  $X$  be a set, and let  $k$  be a natural number. Denote by  $[X]^k$  the set of all  $k$ -subsets of  $X$ , that is, all (unordered !) subsets containing exactly  $k$  elements.

**INFINITE RAMSEY THEOREM.** *Let  $X$  be an infinite set, and let  $k$  be a natural number. For every finite colouring of  $[X]^k$  there exists an infinite subset  $A \subseteq X$  such that the set  $[A]^k$  is monochromatic.*

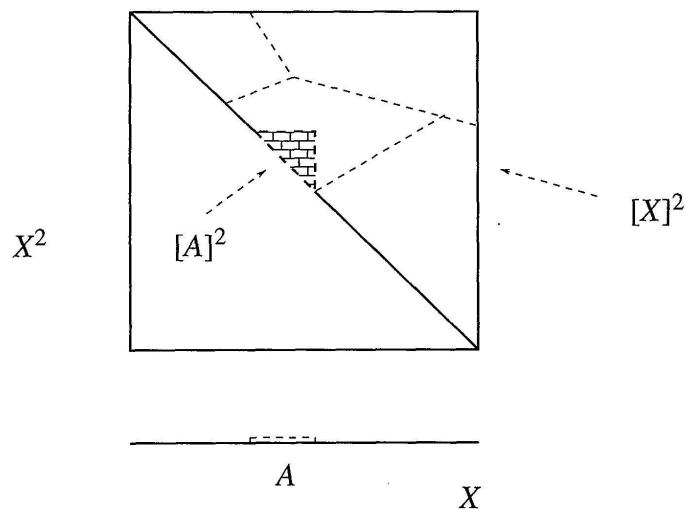


FIGURE 4  
Ramsey theorem for  $k = 2$