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DEFINITION 11 [Gr1]. Say that a G-space X (in our agreed sense) has the *Ramsey–Dvoretzky–Milman property* if for every bounded uniformly continuous function f from X to a finite-dimensional Euclidean space, every $\varepsilon > 0$, and every finite $F \subseteq X$, there is a $g \in G$ with the property

$$\mathrm{Osc}(f|_{gF}) < \varepsilon .$$

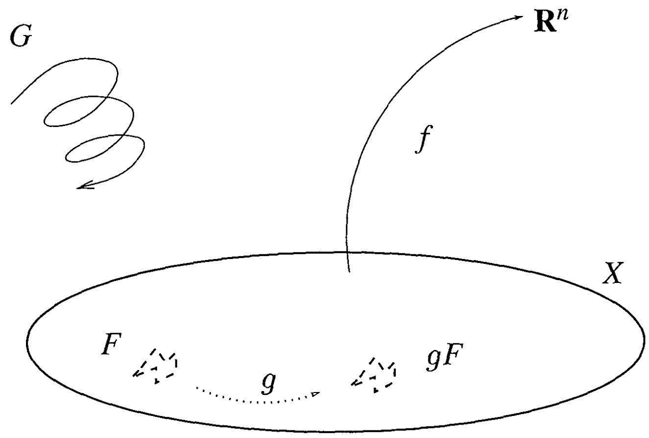

FIGURE 3
The Ramsey–Dvoretzky–Milman property

REMARK 9. Equivalently, F can be assumed compact.

COROLLARY 7. *For a topological group G the following are equivalent:*
(i) *G is extremely amenable;*
(ii) *every metric space X upon which G acts continuously and transitively by isometries has the R–D–M property;*
(iii) *every homogeneous factor-space G/H, equipped with a left-invariant metric (or the left uniform structure), has the R–D–M property.*

Next, we will discover two very important situations where the R–D–M property appears naturally.

6.2 DVORETZKY'S THEOREM

Here is the famous result.

THEOREM (Arieh Dvoretzky). *For all $\varepsilon > 0$ there is a constant $c = c(\varepsilon) > 0$ such that for any n-dimensional normed space $(X, \|\cdot\|_E)$ there is a subspace V of $\dim V \geq c \log n$ and a Euclidean norm $\|\cdot\|_2$ with $\|x\|_2 \leq \|x\|_E \leq (1+\varepsilon)\|x\|_2$ for all $x \in V$.*

The studies of the phenomenon of concentration of measure were given a boost by Vitali Milman's new proof of the Dvoretzky theorem [M1], based on a suitable finite-dimensional approximation to the lemma which follows directly from results that we have previously stated:

LEMMA (Milman). *The pair* $(\mathrm{U}(\mathcal{H}), \mathbf{S}^\infty)$ *has the R–D–M property, where* $\mathbf{S}^\infty$ *is the unit sphere of an infinite-dimensional Hilbert space* $\mathcal{H}$.

6.3 RAMSEY'S THEOREM

Let r be a positive natural number. By $[r]$ one denotes the set $\{1, 2, \ldots, r\}$. A *colouring* of a set X with r colours, or simply *r-colouring,* is any map $\chi\colon X \to [r]$. A subset $A \subseteq X$ is *monochromatic* if for every $a, b \in A$ one has $\chi(a) = \chi(b)$.

Put otherwise, a finite colouring of a set X is nothing but a partition of X into finitely many (disjoint) subsets.

Let X be a set, and let k be a natural number. Denote by $[X]^k$ the set of all k-subsets of X, that is, all (unordered !) subsets containing exactly k elements.

INFINITE RAMSEY THEOREM. *Let* X *be an infinite set, and let* k *be a natural number. For every finite colouring of* $[X]^k$ *there exists an infinite subset* $A \subseteq X$ *such that the set* $[A]^k$ *is monochromatic.*

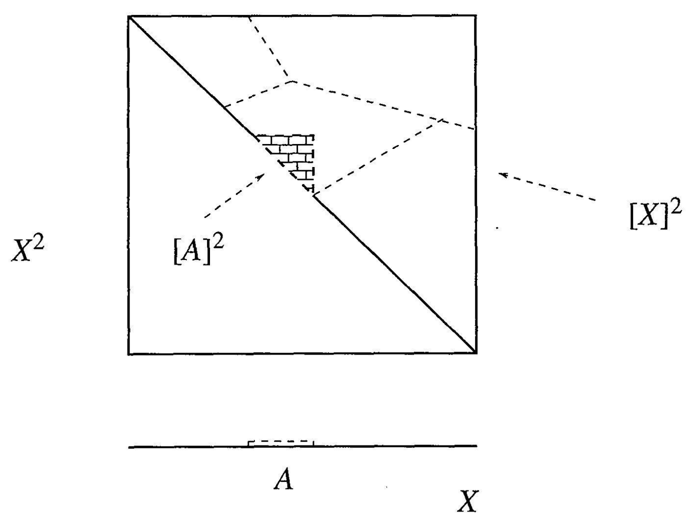

FIGURE 4
Ramsey theorem for $k = 2$