6. RAMSEY–DVORETZKY–MILMAN PROPERTY

6.1 EXTREME AMENABILITY AND SMALL OSCILLATIONS

One way to intuitively describe a 'Ramsey-type result' is as follows. Suppose $\mathfrak{X}$ is a large (and often highly homogeneous) structure of some sort or other. Let $\mathfrak{X}$ be partitioned into a finite number of pieces in an arbitrary way. No matter how irregular and 'ragged' the pieces are, at least one of them always contains the remnants of the original structure, that is, a (possibly much smaller, but still detectable) substructure of the same type which survived intact.

We are now going to link explicitly the fixed point on compacta property to Ramsey-type results. Here is the first step.

EXERCISE 8. Prove that a topological group G is extremely amenable if and only if for every finite collection $g_1,\ldots,g_n$ of elements of G, every bounded right uniformly continuous function $f\colon G\to\mathbf{R}^N$ from G to a finite-dimensional Euclidean space, and every $\varepsilon>0$ there is an $h\in G$ such that $|f(h)-f(g_ih)|<\varepsilon$ for each $i=1,2,\ldots,n$.

[*Hints.* ($\Rightarrow$) The action of G on the space $\mathcal{S}(G)$ of maximal ideals of the C^*-algebra RUCB(G) is continuous, and G itself can be thought of as an everywhere dense subset of $\mathcal{S}(G)$.

($\Leftarrow$) Form a net of suitably indexed elements h as above and consider any limit point of the net $h_\alpha\cdot\xi$, where ξ is an arbitrary element of the compact space upon which G acts continuously.]

EXERCISE 9. Prove that the above condition for extreme amenability is, in turn, equivalent to the following. For every bounded *left* uniformly continuous function f from G to a finite-dimensional Euclidean space, every finite subset F of G, and every $\varepsilon>0$, the oscillation of f on a suitable left translate of F is less than ε :

$$\exists g\in G,\ \mathrm{Osc}(f|_{gF})<\varepsilon\,.$$

It is convenient to deal with the above property in a more general context of G-spaces.

DEFINITION 11 [Gr1]. Say that a G-space X (in our agreed sense) has the *Ramsey–Dvoretzky–Milman property* if for every bounded uniformly continuous function f from X to a finite-dimensional Euclidean space, every $\varepsilon > 0$, and every finite $F \subseteq X$, there is a $g \in G$ with the property

$$\mathrm{Osc}(f|_{gF}) < \varepsilon .$$

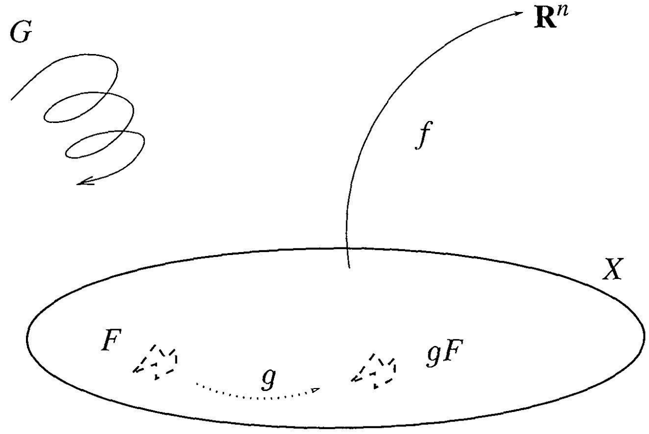

FIGURE 3
The Ramsey–Dvoretzky–Milman property

REMARK 9. Equivalently, F can be assumed compact.

COROLLARY 7. *For a topological group G the following are equivalent:*

(i) *G is extremely amenable;*

(ii) *every metric space X upon which G acts continuously and transitively by isometries has the R–D–M property;*

(iii) *every homogeneous factor-space G/H, equipped with a left-invariant metric (or the left uniform structure), has the R–D–M property.*

Next, we will discover two very important situations where the R–D–M property appears naturally.

6.2 DVORETZKY'S THEOREM

Here is the famous result.

THEOREM (Arieh Dvoretzky). *For all $\varepsilon > 0$ there is a constant $c = c(\varepsilon) > 0$ such that for any n-dimensional normed space $(X, \|\cdot\|_E)$ there is a subspace V of $\dim V \geq c \log n$ and a Euclidean norm $\|\cdot\|_2$ with $\|x\|_2 \leq \|x\|_E \leq (1+\varepsilon)\|x\|_2$ for all $x \in V$.*