

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 48 (2002)  
**Heft:** 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** MM-SPACES AND GROUP ACTIONS  
**Autor:** Pestov, Vladimir  
**Kapitel:** 6.1 EXTREME AMENABILITY AND SMALL OSCILLATIONS  
**DOI:** <https://doi.org/10.5169/seals-66074>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 17.04.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## 6. RAMSEY–DVORETZKY–MILMAN PROPERTY

## 6.1 EXTREME AMENABILITY AND SMALL OSCILLATIONS

One way to intuitively describe a ‘Ramsey-type result’ is as follows. Suppose  $\mathfrak{X}$  is a large (and often highly homogeneous) structure of some sort or other. Let  $\mathfrak{X}$  be partitioned into a finite number of pieces in an arbitrary way. No matter how irregular and ‘ragged’ the pieces are, at least one of them always contains the remnants of the original structure, that is, a (possibly much smaller, but still detectable) substructure of the same type which survived intact.

We are now going to link explicitly the fixed point on compacta property to Ramsey-type results. Here is the first step.

EXERCISE 8. Prove that a topological group  $G$  is extremely amenable if and only if for every finite collection  $g_1, \dots, g_n$  of elements of  $G$ , every bounded right uniformly continuous function  $f: G \rightarrow \mathbf{R}^N$  from  $G$  to a finite-dimensional Euclidean space, and every  $\varepsilon > 0$  there is an  $h \in G$  such that  $|f(h) - f(g_i h)| < \varepsilon$  for each  $i = 1, 2, \dots, n$ .

[Hints. ( $\Rightarrow$ ) The action of  $G$  on the space  $\mathcal{S}(G)$  of maximal ideals of the  $C^*$ -algebra  $\text{RUCB}(G)$  is continuous, and  $G$  itself can be thought of as an everywhere dense subset of  $\mathcal{S}(G)$ .

( $\Leftarrow$ ) Form a net of suitably indexed elements  $h$  as above and consider any limit point of the net  $h_\alpha \cdot \xi$ , where  $\xi$  is an arbitrary element of the compact space upon which  $G$  acts continuously.]

EXERCISE 9. Prove that the above condition for extreme amenability is, in turn, equivalent to the following. For every bounded *left* uniformly continuous function  $f$  from  $G$  to a finite-dimensional Euclidean space, every finite subset  $F$  of  $G$ , and every  $\varepsilon > 0$ , the oscillation of  $f$  on a suitable left translate of  $F$  is less than  $\varepsilon$ :

$$\exists g \in G, \text{Osc}(f|_{gF}) < \varepsilon.$$

It is convenient to deal with the above property in a more general context of  $G$ -spaces.

DEFINITION 11 [Gr1]. Say that a  $G$ -space  $X$  (in our agreed sense) has the *Ramsey–Dvoretzky–Milman property* if for every bounded uniformly continuous function  $f$  from  $X$  to a finite-dimensional Euclidean space, every  $\varepsilon > 0$ , and every finite  $F \subseteq X$ , there is a  $g \in G$  with the property

$$\text{Osc}(f|_{gF}) < \varepsilon.$$

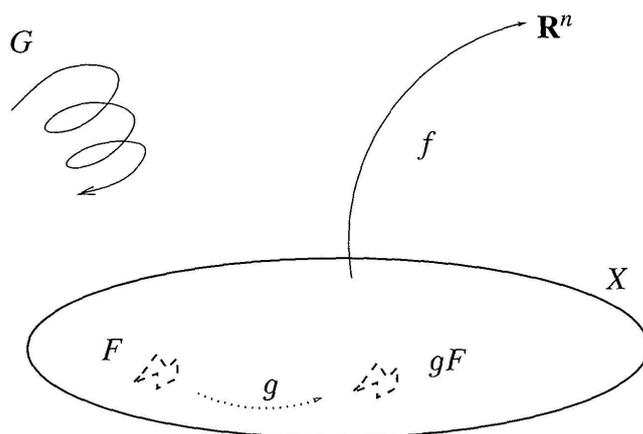


FIGURE 3

The Ramsey–Dvoretzky–Milman property

REMARK 9. Equivalently,  $F$  can be assumed compact.

COROLLARY 7. For a topological group  $G$  the following are equivalent:

- (i)  $G$  is extremely amenable;
- (ii) every metric space  $X$  upon which  $G$  acts continuously and transitively by isometries has the R–D–M property;
- (iii) every homogeneous factor-space  $G/H$ , equipped with a left-invariant metric (or the left uniform structure), has the R–D–M property.

Next, we will discover two very important situations where the R–D–M property appears naturally.

## 6.2 DVORETZKY'S THEOREM

Here is the famous result.

THEOREM (Arieh Dvoretzky). For all  $\varepsilon > 0$  there is a constant  $c = c(\varepsilon) > 0$  such that for any  $n$ -dimensional normed space  $(X, \|\cdot\|_E)$  there is a subspace  $V$  of  $\dim V \geq c \log n$  and a Euclidean norm  $\|\cdot\|_2$  with  $\|x\|_2 \leq \|x\|_E \leq (1 + \varepsilon)\|x\|_2$  for all  $x \in V$ .