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subgroups into a Lévy family. A similar result holds for the group  $\text{Aut}^*(X, \mu)$  of all measure class preserving transformations (Thierry Giordano and the author [G-P]).

## 5. INVARIANT MEANS ON SPHERES

Let a group  $G$  act on a metric space  $X$  by uniform isomorphisms. The formula

$${}^g f(x) = f(g^{-1} \cdot x)$$

determines an action of  $G$  on the space  $\text{UCB}(X)$  of all uniformly continuous bounded complex valued functions on  $X$  by linear isometries. If  $G$  is a topological group acting on  $X$  continuously, the above action of  $G$  on  $\text{UCB}(X)$  need not, in general, be continuous. (An example:  $G = \text{U}(\ell_2)_s$ ,  $X = \mathbf{S}^\infty$ .) However, the action will be continuous if  $X$  is compact. (An easy check.) To some extent, the latter observation can be inverted.

**EXERCISE 7.** Let a topological group  $G$  act continuously on a commutative unital  $C^*$ -algebra  $A$  by automorphisms. Then this action determines a continuous action of  $G$  on the space of maximal ideals of  $A$ , equipped with the usual (weak\*) topology.

Recall that a *mean* on a space  $\mathcal{F}$  of functions is a positive linear functional,  $m$ , of norm one, sending the function 1 to 1. A mean is *multiplicative* if  $\mathcal{F}$  is an algebra and the mean is a homomorphism of this algebra to  $\mathbf{C}$ .

**COROLLARY 2.** Let  $(G, X)$  be a Lévy  $G$ -space. Then there exists a  $G$ -invariant multiplicative mean on the space  $\text{UCB}(X)$  of all bounded uniformly continuous functions on  $X$ .

*Proof.* According to Exercise 7, the group  $G$  acts continuously on the space  $\mathfrak{M}$  of maximal ideals of the  $C^*$ -algebra  $\text{UCB}(X)$ . Therefore,  $\mathfrak{M}$  is an equivariant compactification of  $X$ . By Theorem 4, there is a fixed point  $\varphi \in \mathfrak{M}$ , which is the desired invariant multiplicative mean.  $\square$

The following is deduced by considering Example 11.

COROLLARY 3 [Gr-M1]. *If a compact group  $G$  is represented by unitary operators in an infinite-dimensional Hilbert space  $\mathcal{H}$ , then there exists a  $G$ -invariant multiplicative mean on the uniformly continuous bounded functions on the unit sphere of  $\mathcal{H}$ .*

REMARK 8. The infinite-dimensionality of  $\mathcal{H}$  is essential. Since the unit sphere  $\mathbf{S}$  of a finite-dimensional space  $\mathcal{H}$  is compact, an invariant multiplicative mean on  $\text{UCB}(\mathbf{S})$  exists if and only if there is a fixed vector  $\xi \in \mathbf{S}$ .

Means on  $\text{UCB}(X)$ , where  $X = \mathbf{S}^\infty$  is the unit sphere in the Hilbert space, as well as some other infinite-dimensional manifolds, were studied by Paul Lévy, who viewed them as (substitutes for) infinite-dimensional integrals<sup>4</sup>). The invariant means can thus serve as a substitute for invariant integration on the infinite-dimensional spheres. One can substantially generalize Corollary 3. With this purpose in view, it is convenient to enlarge the concept of a Lévy transformation group.

If  $\mu_1, \mu_2$  are probability measures on the same metric space  $X$ , then the *transportation distance* between them is defined as

$$d_{\text{tran}}(\mu_1, \mu_2) = \inf \int_{X \times X} d(x, y) d\nu(x, y),$$

where the infimum is taken over all probability measures  $\nu$  on the product space  $X \times X$  such that  $(\pi_i)_* \nu = \mu_i$  for  $i = 1, 2$  and  $\pi_1, \pi_2: X \times X \rightarrow X$  denote the coordinate projections.

The way to think of the transportation distance is to identify each probability measure with a pile of sand, then  $d_{\text{tran}}(\mu_1, \mu_2)$  is the minimal average distance that each grain of sand has to travel when the first pile is being moved to take the place of the second<sup>5</sup>).

Let us from now on replace Definition 6 with the following, more general one.

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<sup>4</sup>) The multiplicativity of some of those means, which is not exactly a property one expects of an integral, becomes clear if one recalls an equivalent way to express the concentration phenomenon: on a high-dimensional structure, every 1-Lipschitz function is, probabilistically, almost constant, cf. Section 7.

<sup>5</sup>) In computer science, the transportation distance is known as the Earth Mover's Distance (EMD).

DEFINITION 9. Say that a  $G$ -space  $(G, X)$  is Lévy if there is a net of probability measures  $(\mu_\alpha)$  on  $X$ , such that the  $mm$ -spaces  $(X, d, \mu_\alpha)$  form a Lévy family and for each  $g \in G$ ,

$$d_{\text{tran}}(\mu_\alpha, g\mu_\alpha) \rightarrow 0.$$

Theorems 3 and 4 remain true, with very minor modifications of the proofs.

Here is one application. A unitary representation  $\pi$  of a group  $G$  in a Hilbert space  $\mathcal{H}$  is *amenable* in the sense of Bekka [Be] if there exists a state,  $\varphi$ , on the algebra  $\mathcal{B}(\mathcal{H})$  of all bounded operators on the space  $\mathcal{H}$  of representation, which is invariant under the action of  $G$  by inner automorphisms:  $\varphi(\pi_g T \pi_g^*) = \varphi(T)$  for every  $T \in \mathcal{B}(\mathcal{H})$  and every  $g \in G$ .

THEOREM 5 [P2]. *Let  $\pi$  be a unitary representation of a group  $G$  in a Hilbert space  $\mathcal{H}$ . The following are equivalent.*

- (i)  $\pi$  is amenable.
- (ii) Either  $\pi$  has a finite-dimensional subrepresentation, or  $(G, \mathbf{S})$  has the concentration property (or both).
- (iii) There is a  $G$ -invariant mean on the space  $\text{UCB}(\mathbf{S})$  (a ‘Lévy-type integral’).

*Proof.* (i)  $\Rightarrow$  (ii): according to Th. 6.2 and Remark 1.2.(iv) in [Be], a representation  $\pi$  is amenable if and only if for every finite set  $g_1, g_2, \dots, g_k$  of elements of  $G$  and every  $\varepsilon > 0$  there is a projection  $P$  of finite rank such that for all  $i = 1, 2, \dots, k$

$$\|P - \pi_{g_i} P \pi_{g_i}^*\|_1 < \varepsilon \|P\|_1,$$

where  $\|\cdot\|_1$  denotes the trace class operator norm. It follows that the transportation distance between the Haar measure on the unit sphere in the range of the projection  $P$  and the translates of this measure by operators  $\pi_{g_i}$  can be made as small as desired via a suitable choice of  $P$ . Now a variant of Theorem 4 applies. (See [P2] for details.)

(ii)  $\Rightarrow$  (iii): in the first case, the mean is obtained by invariant integration on the finite-dimensional sphere, while in the second case even a multiplicative mean exists.

(iii)  $\Rightarrow$  (i): let  $\psi$  be a  $G$ -invariant mean on  $\text{UCB}(\mathbf{S}_{\mathcal{H}})$ . For every bounded linear operator  $T$  on  $\mathcal{H}$  define a (Lipschitz) function  $f_T: \mathbf{S}_{\mathcal{H}} \rightarrow \mathbb{C}$  by

$$\mathbf{S}_{\mathcal{H}} \ni \xi \mapsto f_T(\xi) := \langle T\xi, \xi \rangle \in \mathbb{C},$$

and set  $\varphi(T) := \psi(f_T)$ . This  $\varphi$  is a  $G$ -invariant mean on  $\mathcal{B}(\mathcal{H})$ .  $\square$

COROLLARY 4. *A locally compact group  $G$  is amenable if and only if for every strongly continuous unitary representation of  $G$  in an infinite-dimensional Hilbert space the pair  $(G, \mathbf{S}^\infty)$  has the property of concentration.*

COROLLARY 5. *There is no invariant mean on  $\text{UCB}(\mathbf{S}^\infty)$  for the full unitary group  $\text{U}(\ell_2)$ .*

*Proof.* If such a mean existed, then every unitary representation of every group would be amenable, in particular every group would be amenable (by Th. 2.2 in [Be]).

(Of course Corollary 5 also follows from Imre Leader's Example 12 modulo Theorem 2 and Lemma 1.)

A (not necessarily locally compact) topological group  $G$  is *amenable* if there is a left-invariant mean on the space  $\text{RUCB}(G)$  of all right uniformly continuous bounded functions on  $G$ . Denote by  $\text{U}(\ell_2)_u$  the full unitary group with the uniform operator topology.

COROLLARY 6 (Pierre de la Harpe [dlH], proved by different means). *The topological group  $\text{U}(\ell_2)_u$  is not amenable.*

*Proof.* Choose an arbitrary  $\xi \in \mathbf{S}^\infty$ . To every function  $\psi \in \text{UCB}(\mathbf{S}^\infty)$  associate the function  $\tilde{\psi}$  as follows:

$$G \ni g \mapsto \tilde{\psi}(g) := \psi(\pi_g^*(\xi)) \in \mathbf{C}.$$

The correspondence  $\psi \mapsto \tilde{\psi}$  is a  $G$ -equivariant positive bounded unit-preserving linear operator from  $\text{UCB}(\mathbf{S}^\infty)$  to  $\text{RUCB}(\text{U}(\ell_2)_u)$ , and any left-invariant mean  $\varphi$  on the latter  $G$ -module would thus determine a  $G$ -invariant mean on the former  $G$ -module, contradicting Corollary 5.  $\square$

EXAMPLE 13. In a similar fashion, by considering the action of  $\text{Aut}(X, \mu)$  on  $L_0^2(X, \mu)$ , where  $X = \text{SL}(3, \mathbf{R})/\text{SL}(3, \mathbf{Z})$ , one deduces that  $\text{Aut}(X, \mu)_u$  with the uniform topology is not amenable [G-P].