Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 48 (2002)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: MM-SPACES AND GROUP ACTIONS
Autor: Pestov, Vladimir
Kapitel: 5. Invariant means on spheres
DOI: https://doi.org/10.5169/seals-66074

subgroups into a Lévy family. A similar result holds for the group $\mathrm{Aut}^*(X,\mu)$ of all measure class preserving transformations (Thierry Giordano and the author [G-P]).

5. Invariant means on spheres

Let a group G act on a metric space X by uniform isomorphisms. The formula

$$^{g}f(x) = f(g^{-1}\cdot x)$$

determines an action of G on the space $\mathrm{UCB}(X)$ of all uniformly continuous bounded complex valued functions on X by linear isometries. If G is a topological group acting on X continuously, the above action of G on $\mathrm{UCB}(X)$ need not, in general, be continuous. (An example: $G = \mathrm{U}(\ell_2)_s$, $X = \mathbf{S}^{\infty}$.) However, the action will be continuous if X is compact. (An easy check.) To some extent, the latter observation can be inverted.

EXERCISE 7. Let a topological group G act continuously on a commutative unital C^*-algebra A by automorphisms. Then this action determines a continuous action of G on the space of maximal ideals of A, equipped with the usual (weak*) topology.

Recall that a *mean* on a space $\mathcal{F}$ of functions is a positive linear functional, m, of norm one, sending the function 1 to 1. A mean is *multiplicative* if $\mathcal{F}$ is an algebra and the mean is a homomorphism of this algebra to $\mathbf{C}$.

COROLLARY 2. *Let (G,X) be a Lévy G-space. Then there exists a G-invariant multiplicative mean on the space* $\mathrm{UCB}(X)$ *of all bounded uniformly continuous functions on X.*

Proof. According to Exercise 7, the group G acts continuously on the space $\mathfrak{M}$ of maximal ideals of the C^*-algebra $\mathrm{UCB}(X)$. Therefore, $\mathfrak{M}$ is an equivariant compactification of X. By Theorem 4, there is a fixed point $\varphi \in \mathfrak{M}$, which is the desired invariant multiplicative mean. □

The following is deduced by considering Example 11.

COROLLARY 3 [Gr-M1]. *If a compact group G is represented by unitary operators in an infinite-dimensional Hilbert space $\mathcal{H}$, then there exists a G-invariant multiplicative mean on the uniformly continuous bounded functions on the unit sphere of $\mathcal{H}$.*

REMARK 8. The infinite-dimensionality of $\mathcal{H}$ is essential. Since the unit sphere $\mathbf{S}$ of a finite-dimensional space $\mathcal{H}$ is compact, an invariant multiplicative mean on UCB($\mathbf{S}$) exists if and only if there is a fixed vector $\xi \in \mathbf{S}$.

Means on UCB(X), where $X = \mathbf{S}^{\infty}$ is the unit sphere in the Hilbert space, as well as some other infinite-dimensional manifolds, were studied by Paul Lévy, who viewed them as (substitutes for) infinite-dimensional integrals[4]). The invariant means can thus serve as a substitute for invariant integration on the infinite-dimensional spheres. One can substantially generalize Corollary 3. With this purpose in view, it is convenient to enlarge the concept of a Lévy transformation group.

If μ_1, μ_2 are probability measures on the same metric space X, then the *transportation distance* between them is defined as

$$d_{tran}(\mu_1, \mu_2) = \inf \int_{X \times X} d(x, y)\, d\nu(x, y)\,,$$

where the infimum is taken over all probability measures ν on the product space $X \times X$ such that $(\pi_i)_* \nu = \mu_i$ for $i = 1, 2$ and $\pi_1, \pi_2 \colon X \times X \to X$ denote the coordinate projections.

The way to think of the transportation distance is to identify each probability measure with a pile of sand, then $d_{tran}(\mu_1, \mu_2)$ is the minimal average distance that each grain of sand has to travel when the first pile is being moved to take the place of the second[5]).

Let us from now on replace Definition 6 with the following, more general one.

---

[4]) The multiplicativity of some of those means, which is not exactly a property one expects of an integral, becomes clear if one recalls an equivalent way to express the concentration phenomenon: on a high-dimensional structure, every 1-Lipschitz function is, probabilistically, almost constant, cf. Section 7.

[5]) In computer science, the transportation distance is known as the Earth Mover's Distance (EMD).

DEFINITION 9. Say that a G-space (G, X) is Lévy if there is a net of probability measures (μ_α) on X, such that the *mm*-spaces (X, d, μ_α) form a Lévy family and for each $g \in G$,

$$d_{tran}(\mu_\alpha, g\mu_\alpha) \to 0 .$$

Theorems 3 and 4 remain true, with very minor modifications of the proofs.

Here is one application. A unitary representation π of a group G in a Hilbert space $\mathcal{H}$ is *amenable* in the sense of Bekka [Be] if there exists a state, φ, on the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on the space $\mathcal{H}$ of representation, which is invariant under the action of G by inner automorphisms: $\varphi(\pi_g T \pi_g^*) = \varphi(T)$ for every $T \in B(\mathcal{H})$ and every $g \in G$.

THEOREM 5 [P2]. *Let π be a unitary representation of a group G in a Hilbert space $\mathcal{H}$. The following are equivalent.*

(i) *π is amenable.*

(ii) *Either π has a finite-dimensional subrepresentation, or $(G, \mathbf{S})$ has the concentration property (or both).*

(iii) *There is a G-invariant mean on the space* UCB($\mathbf{S}$) *(a 'Lévy-type integral').*

Proof. (i) $\Rightarrow$ (ii): according to Th. 6.2 and Remark 1.2.(iv) in [Be], a representation π is amenable if and only if for every finite set $g_1, g_2, \ldots, g_k$ of elements of G and every $\varepsilon > 0$ there is a projection P of finite rank such that for all $i = 1, 2, \ldots, k$

$$\left\| P - \pi_{g_i} P \pi_{g_i}^* \right\|_1 < \varepsilon \|P\|_1 ,$$

where $\|\cdot\|_1$ denotes the trace class operator norm. It follows that the transportation distance between the Haar measure on the unit sphere in the range of the projection P and the translates of this measure by operators π_{g_i} can be made as small as desired via a suitable choice of P. Now a variant of Theorem 4 applies. (See [P2] for details.)

(ii) $\Rightarrow$ (iii): in the first case, the mean is obtained by invariant integration on the finite-dimensional sphere, while in the second case even a multiplicative mean exists.

(iii) $\Rightarrow$ (i): let ψ be a G-invariant mean on UCB($\mathbf{S}_\mathcal{H}$). For every bounded linear operator T on $\mathcal{H}$ define a (Lipschitz) function $f_T \colon \mathbf{S}_\mathcal{H} \to \mathbf{C}$ by

$$\mathbf{S}_\mathcal{H} \ni \xi \mapsto f_T(\xi) := \langle T\xi, \xi \rangle \in \mathbf{C} ,$$

and set $\varphi(T) := \psi(f_T)$. This φ is a G-invariant mean on $\mathcal{B}(\mathcal{H})$. $\square$

COROLLARY 4. *A locally compact group G is amenable if and only if for every strongly continuous unitary representation of G in an infinite-dimensional Hilbert space the pair $(G, \mathbf{S}^\infty)$ has the property of concentration.*

COROLLARY 5. *There is no invariant mean on* $\mathrm{UCB}(\mathbf{S}^\infty)$ *for the full unitary group* $\mathrm{U}(\ell_2)$.

Proof. If such a mean existed, then every unitary representation of every group would be amenable, in particular every group would be amenable (by Th. 2.2 in [Be]).

(Of course Corollary 5 also follows from Imre Leader's Example 12 modulo Theorem 2 and Lemma 1.)

A (not necessarily locally compact) topological group G is *amenable* if there is a left-invariant mean on the space $\mathrm{RUCB}(G)$ of all right uniformly continuous bounded functions on G. Denote by $\mathrm{U}(\ell_2)_u$ the full unitary group with the uniform operator topology.

COROLLARY 6 (Pierre de la Harpe [dlH], proved by different means). *The topological group* $\mathrm{U}(\ell_2)_u$ *is not amenable.*

Proof. Choose an arbitrary $\xi \in \mathbf{S}^\infty$. To every function $\psi \in \mathrm{UCB}(\mathbf{S}^\infty)$ associate the function $\widetilde{\psi}$ as follows:

$$G \ni g \mapsto \widetilde{\psi}(g) := \psi(\pi_g^*(\xi)) \in \mathbf{C}.$$

The correspondence $\psi \mapsto \widetilde{\psi}$ is a G-equivariant positive bounded unit-preserving linear operator from $\mathrm{UCB}(\mathbf{S}^\infty)$ to $\mathrm{RUCB}(\mathrm{U}(\ell_2)_u)$, and any left-invariant mean φ on the latter G-module would thus determine a G-invariant mean on the former G-module, contradicting Corollary 5. □

EXAMPLE 13. In a similar fashion, by considering the action of $\mathrm{Aut}(X, \mu)$ on $L_0^2(X, \mu)$, where $X = \mathrm{SL}(3, \mathbf{R}) / \mathrm{SL}(3, \mathbf{Z})$, one deduces that $\mathrm{Aut}(X, \mu)_u$ with the uniform topology is not amenable [G-P].