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REMARK 2. In Example 4, replace $\{0,1\}$ with any probability measure space, $X = (X,\mu)$. Equip every finite power X^n with the product measure $\mu^{\otimes n}$ and the normalized Hamming distance $d(x,y) = \frac{1}{n}|\{i\colon x_i \neq y_i\}|$. Unless X is purely atomic, the measures $\mu^{\otimes n}$ are not Borel, and thus X^n aren't even mm-spaces in the sense of our definition. At the same time, if in the definition of the concentration function we only restrict ourselves to measurable subsets A such that A_ε are also measurable, it can be shown that $X^n, n \in \mathbf{N}$ form a Lévy family in a very reasonable sense. (See [Ta1,Ta3] for far-reaching variations.) If anything, this shows that the full formalization of the subject has not yet been achieved and nothing is cast in stone.

Notice that the mm-spaces from the above Examples 1–4 are at the same time (phase spaces of) topological transformation groups, with both metrics and measures being invariant under group actions. In Example 1 it is the action of the orthogonal — or the unitary — group on the sphere, while in Examples 2–4 the groups act upon themselves on the left.

3. A TRANSFORMATION GROUP FRAMEWORK

Here is the idea of what kind of interaction between concentration phenomenon and group actions one should expect. The following example is borrowed from a paper by Vitali Milman [M4].

Suppose a group G acts on an mm-space (X,d,μ) by measure-preserving isometries. Assume that the mm-space X strongly concentrates, that is, the function $\alpha_X(\varepsilon)$ drops off sharply already for small values of ε. Let us assume, for instance, that the concentration is so strong that, whenever $\mu(A) \geq \frac{1}{7}$, the measure of the $\frac{1}{10}$-neighbourhood of A is strictly greater than 0.99. (Cf. Exercise 2.)

If now we partition X into seven pieces, and pick at random one hundred elements $g_1, g_2, \dots, g_{100} \in G$, then at least one of the pieces, say A, has the property that all one hundred translates, of $\frac{1}{10}$-neighbourhoods of A by our elements g_i have a point, x^*, in common. Equivalently, x^* is 'close' (closer than $\frac{1}{10}$) to each of the one hundred translates of A.

The above effect becomes more pronounced the higher the level of concentration is. Partition a concentrated ('high-dimensional') mm-space into a small number of subsets, and at least one of them is hard to move.

In order to set up a formal framework, we assume all topological spaces and topological groups appearing in this article to be metrizable, for the reasons

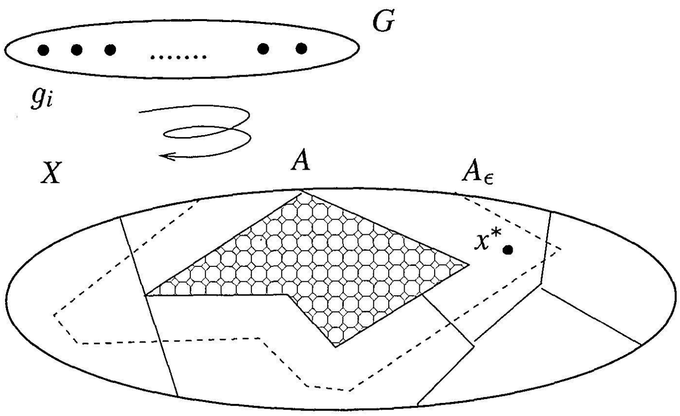

FIGURE 1
Dynamics in the presence of concentration

of mere technical simplicity[2]). We need G-spaces of a particular kind. Let $X = (X,d)$ be a metric space, not necessarily compact, and let a group G act on X (on the left) by *uniformly* continuous maps. In other words, there is a map $G \times X \to X$, $(g,x) \mapsto g \cdot x$, such that $g \cdot (h \cdot x) = (gh) \cdot x$, $e \cdot x = x$, and every map of the form

$$X \ni x \mapsto g \cdot x \in X$$

(a translation by g) is uniformly continuous. (Then it is automatically a uniform isomorphism.) If, moreover, G is a topological group, then we require the action $G \times X \to X$ to be continuous.

EXAMPLE 5. The motivation for our choice of the class of G-spaces is provided by the fact that every (metrizable) compact G-space, K, is such: a translation of K by an element $g \in G$, being a continuous map on a compact space, is uniformly continuous.

Here is another property that compact G-spaces possess automatically, while G-spaces of a more general nature do not.

EXERCISE 3. Let a topological group G act continuously on a (metrizable) compact space $K = (K,d)$. Prove that for every $\varepsilon > 0$ there is a neighbourhood of identity $V \ni e_G$ with the property that whenever $g \in V$ and $x \in K$, one has $d(x, g \cdot x) < \varepsilon$. [In abstract topological dynamics such actions are termed *bounded*, or else *motion equicontinuous*.]

[*Hint.* Using the continuity of the action $G \times K \to K$, choose for each $x \in K$ a neighbourhood U_x of x in K and a neighbourhood, V_x, of e_G in

[2]) More generally, metrics can be replaced with uniform structures.

G, such that $V_x \cdot U_x \subseteq B_\varepsilon(x)$ (the open d-ball around x); now select a finite subcover of $\{U_x\}, \ldots$]

EXAMPLE 6. Every metrizable group admits a right-invariant compatible metric ($d(x,y) = d(xa, ya)$), as well as a left-invariant one ($d(x,y) = d(ax, ay)$). The action of G on itself by left translations is an action by isometries with respect to a left-invariant metric, and (exercise) an action by uniform isomorphisms with respect to a right-invariant metric.

EXERCISE 4. Show that the action of a topological group G upon itself, equipped with a right invariant metric, by left translations, is bounded.

EXAMPLE 7. One topological group of interest to us is $\mathrm{U}(\mathcal{H})_s$, the full unitary group of a separable Hilbert space with the *strong* operator topology. (That is, the topology induced from the Tychonoff product $\mathcal{H}^{\mathcal{H}}$.) A standard neighbourhood of identity in this topology consists of all $T \in \mathrm{U}(\mathcal{H})$ such that $\|T(x_i) - x_i\| < \varepsilon$ for $i = 1, 2, \ldots, n$, where $x_1, \ldots, x_n$ is a finite collection of unit vectors in $\mathcal{H}$. This topology on $\mathrm{U}(\mathcal{H})$ coincides with the *weak operator topology,* that is, the weakest topology making continuous every map of the form

$$\mathrm{U}(\mathcal{H}) \ni T \mapsto \langle x, Tx \rangle \in \mathbf{C}, \;\; x \in \mathcal{H}\,.$$

EXAMPLE 8. Let π be a unitary representation of a group G (viewed as discrete) in a Hilbert space $\mathcal{H}$. Denote by $\mathbf{S}^\infty$ the unit sphere in $\mathcal{H}$, equipped with the norm distance. Then G acts on $\mathbf{S}^\infty$ by isometries: $(g, x) \mapsto \pi_g x$.

REMARK 3. The above G-space is bounded for trivial reasons. It should be noted, however, that in general one does not expect a 'typical' G-space to be bounded at all.

DEFINITION 4. Let a topological group G act continuously, by uniform isomorphisms, on two metric spaces, X and Y. A *morphism*, or an *equivariant map,* from X to Y is a uniformly continuous map $i\colon X \to Y$ which commutes with the action:

$$i(g \cdot x) = g \cdot i(x)\,.$$

DEFINITION 5. Let a topological group G act continuously on a metric space (X, d) by uniformly continuous maps, and let also G act continuously on a compact space K. Let $i \colon X \to K$ be a morphism of G-spaces with an everywhere dense image in K. The pair (K, i) is called an *equivariant compactification* of X.

EXAMPLE 9. Let G and $\mathcal{H}$ be as in Example 8. The unit ball $\mathbf{B}$ in $\mathcal{H}$ equipped with the weak topology is compact, and G acts on $\mathbf{B}$ in the same way as on the sphere. The embedding $\mathbf{S}^\infty \hookrightarrow \mathbf{B}$ is an equivariant compactification.

The following is at the heart of abstract topological dynamics.

THEOREM 1. *Let G be a topological group, and let d be a right-invariant metric generating the topology of G. Let K be a (metric) compact G-space, and let $\kappa \in K$ be arbitrary. There is a morphism of G-spaces $i \colon (G, d) \to K$ such that $i(e) = \kappa$.*

Proof. Define the map $i \colon G \to K$ (an *orbit map*) by

$$i \colon G \ni y \mapsto y \cdot \kappa \in K .$$

This map is equivariant. $[i(g \cdot y) = (gy) \cdot \kappa = g \cdot (y \cdot \kappa) = g \cdot i(y).]$ It only remains to check the uniform continuity of i. Choose any continuous metric on K, say ρ. Using Exercise 3, find a $\delta > 0$ with the property that $\rho(x, g \cdot x) < \varepsilon$ whenever $x \in K$ and $d(g, e_G) < \delta$. If now $g, h \in G$ are such that $d(g, h) < \delta$, then $d(gh^{-1}, e_G) < \delta$ and consequently

$$\rho(h\kappa, g\kappa) = \rho(h\kappa, gh^{-1}(h\kappa)) < \varepsilon . \qquad \square$$

REMARK 4. The difference between the right and left invariant metrics (or, more generally, uniform structures) on a topological group cannot be overemphasized. Even if they are totally symmetric, they cease to be such as soon as we choose the action (in our case, by left translations).

Here is a key notion putting the concentration of measure in a dynamical context.

DEFINITION 6. Let a metrizable topological group G act continuously by uniform isomorphisms on a metric space $X = (X, d)$. Say that the G-space (transformation group) (G, X) is *Lévy* (Gromov and Milman [Gr-M1]) if there are a sequence of subgroups of G

$$G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n \subseteq \cdots \subseteq G,$$

and a sequence of probability measures

$$\mu_1, \mu_2, \ldots, \mu_n, \ldots$$

on (X, d), such that

(i) $\bigcup G_n$ is everywhere dense in G,

(ii) μ_n are G_n-invariant,

(iii) (X, d, μ_n) form a Lévy family.

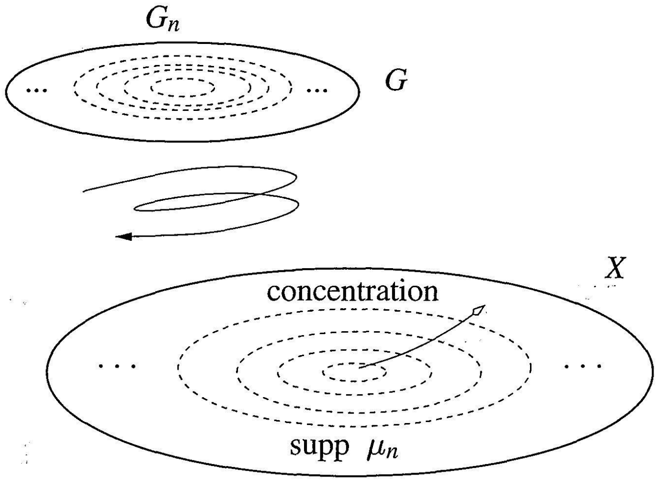

FIGURE 2

A Lévy transformation group

In the particular case where X is the group itself equipped with a right-invariant metric and the action of G is by left translations, we say that G is a *Lévy group.*

EXAMPLE 10. Let $\mathcal{H} = \ell_2$, and let $G = \mathrm{U}(\mathcal{H})_s$, $X = (G, d)$, where d is a right-invariant metric and the action is by left translations. Set $G_n = \mathrm{SU}(n)$ (embedded into $\mathrm{U}(\mathcal{H})$ as a subgroup of block-diagonal operators), and let μ_n denote the normalized Haar measure on $\mathrm{SU}(n)$. One can view μ_n as a measure on all of $\mathrm{U}(\ell_2)_s$ with support $\mathrm{SU}(n)$. The *mm*-spaces $(\mathrm{U}(\mathcal{H})_s, d, \mu_n)$ clearly form a Lévy family, because the spaces $(\mathrm{SU}(n)_s, d|_{\mathrm{SU}(n)}, \mu_n)$ do. We conclude: $\mathrm{U}(\mathcal{H})_s$ is a Lévy group.

EXAMPLE 11. Let π be a strongly continuous unitary representation of a compact group G in ℓ_2. Then ℓ_2 decomposes into the orthogonal direct sum of finite-dimensional (irreducible) unitary G-modules, $\ell_2 \cong \bigoplus_{n=1}^{\infty} V_n$. Set for each $n \in \mathbf{N}$

$$\mathbf{S}_n = \mathbf{S}^\infty \cap \bigoplus_{i=1}^{n} V_n .$$

We obtain a nested sequence of spheres of increasing finite dimension which are invariant under the action of G. Let μ_n denote the rotation-invariant probability measure on the sphere $\mathbf{S}_n$. Denote also $G_n = G$ for all n. Then $(G, \mathbf{S}^\infty)$ is a Lévy transformation group.

4. CONCENTRATION PROPERTY AND FIXED POINTS

The following definition is an attempt to capture 'concentration in the absence of measure' (as indeed there are typically no invariant measures on infinite dimensional spaces).

DEFINITION 7 [M2,M3]. Let a group G act on a metric space X by uniform isomorphisms. Call a subset $A \subseteq X$ *essential* if for every $\varepsilon > 0$ and every finite collection $g_1, \ldots, g_N \in G$ one has

$$\bigcap_{i=1}^{N} g_i A_\varepsilon \neq \varnothing .$$

(Have another look at Fig. 1 !)

EXERCISE 5. The definition obtained by replacing $g_i A_\varepsilon$ with $(g_i A)_\varepsilon$ is equivalent.

Informally speaking, an essential set is so 'big' that translates of any ε-neighbourhood of it, taken in any finite number, don't fit in without overlapping.

DEFINITION 8 (*ibidem*). A G-space X has the *concentration property* if every finite cover of X contains at least one essential set.

Perhaps one gets a better idea of the property if we start with an example where it is violated.