

## 2. Some concepts of asymptotic geometric analysis

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theory, cf. papers by Gromov [Gr1], Milman [M2,M3], and some others [A-M,Gl,P2,P3,G-P,Gl-W]. However, it is safe to say that there is still a long way to go towards the full understanding of the picture.

Here we aim at providing a readable introduction into this circle of ideas.

## 2. SOME CONCEPTS OF ASYMPTOTIC GEOMETRIC ANALYSIS

DEFINITION 1. A space with metric and measure, or an *mm-space*, is a triple  $(X, d, \mu)$ , where  $d$  is a metric on a set  $X$  and  $\mu$  is a finite Borel measure on the metric space  $(X, d)$ . It will be convenient to assume throughout that  $\mu$  is a probability measure, that is, normalized to one.

DEFINITION 2. The *concentration function*  $\alpha_X$  of an *mm-space*  $X = (X, d, \mu)$  is defined for non-negative real  $\varepsilon$  as follows:

$$\alpha_X(\varepsilon) = \begin{cases} \frac{1}{2} & \text{if } \varepsilon = 0, \\ 1 - \inf\{\mu(A_\varepsilon) : A \subseteq X \text{ is Borel, } \mu(A) \geq \frac{1}{2}\} & \text{if } \varepsilon > 0. \end{cases}$$

Here  $A_\varepsilon$  denotes the  $\varepsilon$ -neighbourhood ( $\varepsilon$ -fattening,  $\varepsilon$ -thickening) of  $A$ .

EXERCISE 1. Prove that  $\alpha(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow \infty$ . (For spaces of finite diameter this is of course obvious.)

DEFINITION 3. An infinite family of *mm-spaces*,  $(X_n, d_n, \mu_n)_{n=1}^\infty$ , is called a *Lévy family* if the concentration functions  $\alpha_n$  of  $X_n$  converge to zero pointwise on  $(0, \infty)$ :

$$\forall \varepsilon > 0, \alpha_n(\varepsilon) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

EXERCISE 2. Prove that the above condition is equivalent to the following. Let  $A_n \subseteq X_n$  be Borel subsets with the property that

$$\liminf_{n \rightarrow \infty} \mu_n(A_n) > 0.$$

Then

$$\forall \varepsilon > 0, \lim_{n \rightarrow \infty} \mu_n((A_n)_\varepsilon) = 1.$$

The following are some of the most common examples of Lévy families.

EXAMPLE 1. Unit spheres  $S^n$  in the Euclidean spaces  $\mathbf{R}^{n+1}$ , equipped with the Euclidean (or geodesic) distances and the normalized Haar measures (that is, the unique rotation-invariant probability measures). This result is due to Paul Lévy [Lév], though his proof, based on the isoperimetric inequality, was only made rigorous much later by Gromov [Gr2]. (Nowadays simpler proofs, using the Brunn–Minkowski inequality, are known, cf. [Gr-M2, Sch].)

EXAMPLE 2. The special orthogonal groups  $SO(n)$ , equipped with the normalized Haar measure and the uniform operator metric,

$$d(T, S) := \|T - S\|,$$

induced from  $\mathcal{B}(\mathbf{R}^n) \cong M_n$ . This was established by Gromov and Milman [Gr-M1]. The same argument holds for the special unitary groups.

EXAMPLE 3. The family of finite permutation groups  $(S_n)$ , equipped with the uniform (normalized counting) measure and the Hamming distance:

$$d(\sigma, \tau) = \frac{1}{n} |\{i: \sigma(i) \neq \tau(i)\}|.$$

The result is due to Maurey [Ma], see also [Ta1].

EXAMPLE 4. The Hamming cubes  $\{0, 1\}^n$  equipped with the normalized counting measure and the Hamming distance  $d(x, y) = \frac{1}{n} |\{i: x_i \neq y_i\}|$  form a Lévy family [Sch, M-S].

REMARK 1. All of the above are *normal* Lévy families, meaning that the concentration functions  $\alpha_n$  admit Gaussian upper bounds:

$$\alpha_n(\varepsilon) \leq C_1 \exp(-C_2 n \varepsilon^2)$$

for some  $C_1, C_2 > 0$ .

It should be noted that this is not always the case for ‘naturally occurring’ Lévy families. For instance, the groups  $SL(2, \mathbf{F}_p)$ , where  $p$  are prime numbers, equipped with the normalized counting measure and the word metric given by a fixed system of generators in  $SL(2, \mathbf{Z})$ , form a Lévy family with  $\alpha_p(\varepsilon) \leq C_1 \exp(-C_2 \sqrt{p} \varepsilon)$ , [A-M, M4]. (Recall in this connection that the  $n$ -th prime number  $p_n \sim n \log n$ .)

REMARK 2. In Example 4, replace  $\{0, 1\}$  with any probability measure space,  $X = (X, \mu)$ . Equip every finite power  $X^n$  with the product measure  $\mu^{\otimes n}$  and the normalized Hamming distance  $d(x, y) = \frac{1}{n} |\{i: x_i \neq y_i\}|$ . Unless  $X$  is purely atomic, the measures  $\mu^{\otimes n}$  are not Borel, and thus  $X^n$  aren't even *mm*-spaces in the sense of our definition. At the same time, if in the definition of the concentration function we only restrict ourselves to measurable subsets  $A$  such that  $A_\varepsilon$  are also measurable, it can be shown that  $X^n, n \in \mathbf{N}$  form a Lévy family in a very reasonable sense. (See [Ta1, Ta3] for far-reaching variations.) If anything, this shows that the full formalization of the subject has not yet been achieved and nothing is cast in stone.

Notice that the *mm*-spaces from the above Examples 1–4 are at the same time (phase spaces of) topological transformation groups, with both metrics and measures being invariant under group actions. In Example 1 it is the action of the orthogonal — or the unitary — group on the sphere, while in Examples 2–4 the groups act upon themselves on the left.

### 3. A TRANSFORMATION GROUP FRAMEWORK

Here is the idea of what kind of interaction between concentration phenomenon and group actions one should expect. The following example is borrowed from a paper by Vitali Milman [M4].

Suppose a group  $G$  acts on an *mm*-space  $(X, d, \mu)$  by measure-preserving isometries. Assume that the *mm*-space  $X$  strongly concentrates, that is, the function  $\alpha_X(\varepsilon)$  drops off sharply already for small values of  $\varepsilon$ . Let us assume, for instance, that the concentration is so strong that, whenever  $\mu(A) \geq \frac{1}{7}$ , the measure of the  $\frac{1}{10}$ -neighbourhood of  $A$  is strictly greater than 0.99. (Cf. Exercise 2.)

If now we partition  $X$  into seven pieces, and pick at random one hundred elements  $g_1, g_2, \dots, g_{100} \in G$ , then at least one of the pieces, say  $A$ , has the property that all one hundred translates, of  $\frac{1}{10}$ -neighbourhoods of  $A$  by our elements  $g_i$  have a point,  $x^*$ , in common. Equivalently,  $x^*$  is ‘close’ (closer than  $\frac{1}{10}$ ) to each of the one hundred translates of  $A$ .

The above effect becomes more pronounced the higher the level of concentration is. Partition a concentrated (‘high-dimensional’) *mm*-space into a small number of subsets, and at least one of them is hard to move.

In order to set up a formal framework, we assume all topological spaces and topological groups appearing in this article to be metrizable, for the reasons